Term-Structure of Consumption Risk Premia in the Cross-Section of Currency Returns[*]

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Abstract

I quantify the risk-return relationship in the foreign exchange market in the cross-section and across investment horizons by focusing on the role of multiple sources of US consumption risk. To this end, I estimate a flexible structural model of the joint dynamics of aggregate consumption, inflation, nominal interest rate, and stochastic variance with cross-equation restrictions implied by recursive preferences. I identify the following four structural shocks: inflation, short-run, long-run and variance consumption risks. To measure their relative importance, I compute marginal quantities and prices of risk (marginal Sharpe ratios) in the cross-section of currency baskets for horizons from one quarter to ten years. I find that the long-run consumption risk plays a prominent role: it carries a Sharpe ratio of 0.66 and contributes the most to the level and spread of excess returns between baskets of high and low interest rate currencies at all investment horizons. The short-run consumption risk has an effect at the horizon of one quarter only, where it explains at least 26% of the corresponding spread in excess returns.

Keywords: consumption risks, term-structure of risk, term-structure of risk compensation, cross-section of currencies, shock elasticity.

1 Introduction

In this paper, I quantify the risk-return relationship in the foreign exchange market in the cross-section and across investment horizons. I perform the analysis from the perspective of a US representative agent with recursive preferences over consumption. As in the long-run risk literature, I allow for the possibility that there are multiple sources of risk affecting consumption growth, such as shocks to expected consumption growth, stochastic variance of consumption growth, and consumption growth itself. My focus is on identifying such shocks, measuring their impact on currency prices in the cross-section and at different investment horizons, and understanding their relative importance for multi-period currency risk premia.

An influential paper by Lustig and Verdelhan (2007) shows that sorting currencies by their respective interest rates generates baskets with different exposures to realized consumption growth, which can explain the cross-sectional differences in one-period currency risk premia. The authors limit their attention to a fixed investment horizon that corresponds to the decision interval of the representative agent with recursive preferences. My contribution is in expanding the analysis to alternative horizons and characterizing multiple sources of consumption risk.

My interest lies in describing empirical properties of consumption risks. Therefore, instead of taking a stand on a specific process for consumption growth, I estimate a flexible vector autoregression (VAR) with stochastic variance that captures the spirit of the long-run risk models. I account for stochastic variance because as it has been widely documented in the literature (e.g., Bansal, Kiku, Shaliastovich, and Yaron, 2012; Campbell, Giglio, Polk, and Turley, 2012), the time-variation in the variance of consumption growth has first order implications for the macro dynamics and properties of asset prices.

An important feature of my approach is that I use additional information about the consumption process contained in macro variables and asset prices. An asset price is an appealing source of information about consumption because in equilibrium it reflects the unobservable components of the consumption growth process that are difficult to estimate on the basis of consumption data alone. Specifically, I learn about the consumption growth process through the joint dynamics of consumption growth, inflation, and the nominal yield.

I choose nominal bond as an asset because the nominal yield reflects risks relevant for exchange rates, as the theoretical literature (e.g., Bansal and Shaliastovich, 2012)

has emphasized. In addition, the use of the yield, as opposed to another asset price, is convenient because it does not require the modeling of any cash flow dynamics. I incorporate inflation for two reasons. First, inflation has forecasting ability for future consumption growth (Piazzesi and Schneider, 2006). Second, I need inflation dynamics to convert the model implied real risk-free rate to the observed nominal interest rate.

The pricing kernel derived by applying recursive preferences to the consumption growth process depends on the nominal yield because it is one of the states of the model. On the other hand, the pricing kernel must value all assets, including the nominal yield. The twofold role of the nominal yield in the model implies a set of pricing restrictions on the VAR parameters.

In summary, I specify my model of consumption growth in the form of a vector autoregression with stochastic variance and structural restrictions derived under recursive preferences. This approach is novel, and it adds power to identify expected consumption growth better. I estimate the model by using quarterly data for US consumption growth, inflation, and a three-month nominal yield from the second quarter of 1947 until the fourth quarter of 2011. For estimation, I employ the Bayesian Markov Chain Monte Carlo (MCMC) methods. The key advantage of this approach is that it allows me imposing the required pricing restrictions directly; in addition, this approach delivers the estimated time series of stochastic variance.

I identify structural shocks from the estimated reduced-form innovations choosing to work with globally identified systems. I show that I have a choice of only two systems because of various restrictions based on economic intuition and regularity conditions (Rubio-Ramirez, Waggoner, and Zha, 2010). My model features the following four structural shocks to consumption: (1) the short-run consumption risk, (2) the inflation risk, (3) the long-run consumption risk, and (4) the variance consumption risk. The only difference between the identification schemes is in the underlying identifying assumptions about the short-run consumption and inflation risks. I label the identification schemes "Fast Inflation" and "Fast Consumption". Under "Fast Inflation", inflation reacts to the short-run consumption shock contemporaneously, whereas consumption growth reacts to the inflation shock with a one-quarter delay (inflation is a faster variable). In contrast, under "Fast Consumption", consumption growth reacts to the inflation shock contemporaneously, whereas inflation reacts to the consumption growth shock with a one-quarter delay (consumption growth is a faster variable).

I use data on twelve currencies of developed economies over the period from 1986 to 2011 at a quarterly frequency. The choice of currencies is limited by the availability of

the term-structure data required for the cross-sectional sorting. I sort currencies into three currency baskets based on the level of the average foreign yield.

I find that the model fits the macroeconomic data and data on asset prices well. First, the model captures important economic episodes such as the Great Moderation, recessions, and the recent financial crisis. Second, diagnostics of fitting errors do not exhibit noticeable misspecification. Hence, this provides a realistic setup for examining the model's asset pricing implications. I perform such an analysis across forty investment horizons, from one quarter to ten years. I use the shock-exposure and shock-price elasticities of Borovička, Hansen, Hendricks, and Scheinkman (2011) and Borovička and Hansen (2011) to characterize the term-structure of consumption risks and their prices in the cross-section of currency baskets at alternative horizons. Shock elasticities measure the sensitivity of expected cash flows and returns with respect to the change in the amount of the underlying risk and account for the presence of stochastic variance. The elasticities represent marginal quantities and marginal prices of risk (marginal Sharpe ratios).

I document the prominent role of the long-run consumption risk for currency pricing in the cross-section and across multiple horizons. This is the main finding of my paper. First, there is a stable cross-sectional pattern in the term-structure of quantities of risk. At all horizons, the low interest rate currencies act as a hedge against the long-run risk, whereas the high interest rate currencies display a positive exposure to the risk. Differences are statistically significant and economically meaningful. Second, the long-run risk is associated with the highest risk compensation: its one-period log Sharpe ratio is 0.66. At the horizon of one quarter, the long-run risk explains at least 48% of the cross-sectional spread in excess returns.

The other shocks contribute to risk premia less prominently. The short-run consumption risk is priced in the cross-section of currency baskets at the horizon of one quarter only. This finding is based on the fact that only the contemporaneous difference in exposures of the low and the high interest rate currencies to the short-run risk is statistically significant. At longer horizons, currencies are mostly immune to the short-run risk or their risk exposures are insignificantly different from each other. The short-run consumption risk carries a high one-period log Sharpe ratio of 0.58 (0.52) under the "Fast Inflation" ("Fast Consumption") identification scheme, and, as a result, it explains at least 26% of the spread in one-period excess returns between the corner baskets.

At most investment horizons, currency baskets have significantly different exposures to the inflation risk. Similarly to the case of the long-run risk, low interest rate currencies

act as a hedge against the risk, whereas the high interest rate currencies have a positive exposure to the risk. However, the price of the inflation risk is statistically significant only if consumption growth is a fast variable, i.e., reacts contemporaneously to the inflation risk. In this case, the inflation risk explains 26% of the spread in the one-period excess returns. The contribution of the inflation risk in explaining the cross-sectional spread in excess returns is smaller than the contribution of the long-run risk because its multi-period price of risk (e.g., a one period log Sharpe ratio is 0.26) is less than a half of that for the long-run risk.

Finally, the loadings of FX cash flows on the variance risk are not significantly different in the cross-section at any horizon. The variance risk matters in a different respect. All currency baskets are highly sensitive to the risk at horizons longer than three years: the impact of a positive variance shock gradually increases with time and eventually causes substantial declines in cash flows of all currency baskets. However, the marginal price of this exposure is small. For example, at a three year horizon the marginal Sharpe ratio associated with the low interest rate currencies is 0.07, with the intermediate interest rate currencies is 0.06, and with the high interest rate currencies is 0.08. The marginal prices are positive suggesting that the currency baskets act as marginal hedges against an unfavorable variance shock to the representative agent.

Related literature

My paper is related to two strands of international macro-finance literature that examines time-series and cross-sectional properties of currency risk premia. I limit my discussion to papers that interpret currency risk premia as compensation for consumption risks. On the empirical front, Sarkissian (2003) and Lustig and Verdelhan (2007) study the ability of the consumption growth factor to explain the cross-section of currency returns. Sarkissian (2003) adapts the framework of Constantinides and Duffie (1996) to a multi-country setting and documents that the cross-country variance of consumption growth exhibits explanatory power for cross-sectional differences in returns on individual currencies, whereas the consumption growth itself does not. Lustig and Verdelhan (2007) establish in the framework of the durable CCAPM of Yogo (2006) that the consumption growth is a priced factor in the cross-section of returns on currency baskets formed by sorting currencies by respective interest rates. There are two common features in these papers. First, both studies recognize the presence of multiple sources of consumption risk but do not describe them explicitly. Second, both papers do not extend the analysis beyond a fixed horizon given by a decision interval of the

representative agent (one quarter in the case of Sarkissian, 2003, and one year in the case of Lustig and Verdelhan, 2007).

Part of the theoretical literature features different consumption-based models dedicated to rationalizing the time-series behavior of currency risk premia, e.g., the violation of the uncovered interest rate parity. Models include, but not limited to, settings with habits (Heyerdahl-Larsen, 2012; Verdelhan, 2010), long-run risks (Bansal and Shaliastovich, 2012; Colacito, 2009; Colacito and Croce, 2012), and disasters (Gabaix and Farhi, 2011). My paper is closely related to the international long-run risk literature, but my focus is different. Theoretical international long-run risk studies model a joint distribution of domestic and foreign macroeconomic quantities to pin down a theoretical equilibrium exchange rate consistent with the forward premium anomaly. Instead, I model multiple sources of consumption risk of the US representative agent, estimate them, and establish their relative importance for currency risk premia in the cross-section of currencies and across alternative investment horizons.

My paper is also related to Hansen, Heaton, and Li (2008), who provide evidence on the importance of the permanent shock to consumption growth in account for the value premium. The similarity is in terms of approach, that is establishing the importance of consumption risks for explaining the cross-section of asset prices by joint modeling the stochastic discount factor (under the assumption of recursive preferences) and cash flow processes. My study differs from Hansen, Heaton, and Li (2008) in three principal dimensions. First, I study the foreign exchange market, which has been less examined than the US equity market. Second, my model has stochastic variance, so I account for variation in volatility of consumption growth, and therefore, in risk premia. Third, I quantify the relative importance of consumption risks at short and medium horizons, rather than at infinite horizons.

2   The Model

Similarly to Lustig and Verdelhan (2007), I study the relative importance of consumption risks for currency pricing from the viewpoint of the US representative agent. In other words, I examine how currency cash flows covary with the US consumption risks and how this covariation is priced. My key modeling ingredients are: (1) the stochastic discount factor implied by the preferences of the representative agent and the dynamics of the consumption growth process and (2) currency cash flow. I proceed by describing each component in turn.

2.1 Recursive preferences

I use a standard framework of the representative agent model with recursive preferences. The recursive utility of Epstein and Zin (1989) and Weil (1989) is designed to account for the temporal distribution of risks; therefore, it is a natural setting for studying the role of multiple sources of risk. Notable applications of this framework for understanding the joint dynamics of exchange rates, macro quantities, and asset prices include, but not limited to, Backus, Gavazzoni, Telmer, and Zin (2010), Bansal and Shaliastovich (2012), Colacito and Croce (2011), Colacito and Croce (2012), Colacito (2009), Tretvoll (2011a), and Tretvoll (2011b).

The recursive utility is a constant elasticity of substitution recursion,

$$U_t = [(1-\beta)c_t^\rho + \beta\mu_t(U_{t+1})^\rho]^{1/\rho}, \tag{2.1}$$

with the certainty equivalent function,

$$\mu_t(U_{t+1}) = [E_t(U_{t+1}^\alpha)]^{1/\alpha}, \tag{2.2}$$

where c_t is consumption at time t, U_t is utility from time t onwards, $(1-\alpha)$ is the coefficient of relative risk aversion, $1/(1-\rho)$ is the elasticity of intertemporal substitution (EIS), and β is the subjective discount factor.

Under recursive preferences, the stochastic discount factor $m_{t,t+1}$ has two components, consumption growth and a forward looking component:

$$m_{t,t+1} = \beta(c_{t+1}/c_t)^{\rho-1}(U_{t+1}/\mu_t(U_{t+1}))^{\alpha-\rho}. \tag{2.3}$$

Appendix A.5 of the NBER version of Backus, Chernov, and Zin (2012) provides the derivation. There are two alternative ways to consider the component $U_{t+1}/\mu_t(U_{t+1})$ and to further derive the pricing kernel. One possibility is to use the connection between U_t and the equilibrium value of the aggregate consumption stream. This link would imply that the log stochastic discount factor is a function of consumption growth, $\log g_{t,t+1} = \log(c_{t+1}/c_t)$, and the return to a claim on future wealth, $r_{t,t+1}^w$, (Epstein and Zin, 1991):

$$\log m_{t,t+1} = \alpha/\rho \cdot \log\beta - \alpha(1-\rho)/\rho \cdot \log g_{t,t+1} - (1-\alpha/\rho) \cdot \log r_{t,t+1}^w. \tag{2.4}$$

The other possibility is to specify the process for consumption growth explicitly and derive this component of the pricing kernel as a function of the model's states and fundamental shocks.

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The latter approach serves my purpose of describing the relative importance of multiple sources of consumption risk for currency pricing across multiple horizons. First, under the null of a structural model, the multi-period objects (consumption growth, stochastic discount factor, and cash flow) directly follow from the dynamics of the corresponding one-period objects. Therefore, a multi-period characterization of currency risk exposures and corresponding prices of risk does not require more data than its one-period counterpart. Second, this setting allows the decomposition of the total risk premium into the contributions of different sources of risk across multiple horizons (Borovička and Hansen, 2011).

2.2 Consumption growth process

It is a well known problem in asset pricing that high-quality consumption data are available at low frequency, and consequently, the identification of multiple sources of consumption risk is a challenging task. As a result, most studies of the joint behaviour of macro economic quantities and asset prices are theoretical. Authors calibrate various empirically plausible processes for consumption growth and study the implications of these models for asset prices.

The common critique of this approach is that different consumption growth processes are observationally equivalent given small sample sizes. Nonetheless, they have very different implications for asset prices. This observation has two implications. On the one hand, an econometrician working with consumption based models faces a serious challenge. On the other hand, this observation suggests that theoretical asset prices are informative about the consumption growth process. Indeed, in equilibrium asset prices are functions of observable consumption growth and unobservable states. As a result, one can learn about the data-generating process for consumption growth by observing asset prices. I exploit this implication to identify consumption growth empirically.

I specify a parsimonious yet flexible model of consumption growth. I posit a vector autoregressive process for $Y_{t+1} = (\log g_{t,t+1},\ \log \pi_{t,t+1},\ i^1_{t+1},\ \sigma^2_{t+1})'$ that includes consumption growth $\log g_{t,t+1}$, inflation $\log \pi_{t,t+1}$, short-term nominal yield i^1_{t+1}, and the stochastic variance σ^2_{t+1}

$$Y_{t+1} \;=\; F + GY_t + H\sigma_t\varepsilon_{t+1}, \tag{2.5}$$

where F is a four-by-one column-vector, and G and H are four-by-four matrices.[1] The

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[1] I use double subscripts for log consumption growth and inflation to indicate the time period of the

vector ε_{t+1} contains four structural shocks, $\varepsilon_{t+1} = (\varepsilon_{g,t+1},\ \varepsilon_{\pi,t+1},\ \varepsilon_{i,t+1},\ \varepsilon_{\sigma,t+1})'$. Shocks $\varepsilon_{g,t+1}$, $\varepsilon_{\pi,t+1}$, and $\varepsilon_{\sigma,t+1}$ are the consumption risk, the inflation risk, and the variance risk, respectively. I interpret the fourth shock $\varepsilon_{i,t+1}$ later, once I have obtained the estimation results. I impose six parameter restrictions $G_{41} = G_{42} = G_{43} = H_{41} = H_{42} = H_{43} = 0$ to ensure that the stochastic variance follows the discretized version of the continuous-time square-root process.[2]

I select a variable to be included in Y_t if the variable has forecasting power for the future consumption growth. Hall (1983) and Hansen and Singleton (1983) show that lagged consumption growth is useful in predicting future US consumption growth. Piazzesi and Schneider (2006) argue that inflation is a leading recession indicator. Bansal, Kiku, and Yaron (2012b), Constantinides and Ghosh (2011), and Colacito and Croce (2011) argue that the real risk-free rate serves as a direct measure of the predictable component in future consumption growth. Instead of including the real risk-free rate in Y_t, I use a short-term nominal yield and inflation.[3]

Among the possible asset prices, I use the nominal yield for a number of reasons. First, the extant empirical and theoretical literature on the violation of the uncovered interest rate parity has documented that risks in exchange rates and interest rates are related (e.g., Bansal and Shaliastovich, 2012; Heyerdahl-Larsen, 2012; Verdelhan, 2010). At a later stage in my paper, I project the currency prices on the US stochastic discount factor. Therefore, it is critical to ensure that important sources of exchange rate risks are captured by the model of the stochastic discount factor. Second, the use of the yield does not require modeling of an extra cash flow process, e.g., the dividend process, or taking a stand on whether the stock market return is a good proxy for the return on the aggregate consumption claim.

I introduce stochastic variance to the model because the time-variation in the volatility of consumption growth is a salient feature of consumption data, which in its turn serves

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corresponding change in consumption or price level. For example, $\log \pi_{t,t+\tau}$ is a $\tau-$period inflation from t to $t + \tau$. I use superscripts for interest rates to indicate the type of the corresponding yields. For example, i_t^τ corresponds to the yield of the τ-period nominal bond at time t. σ_t^2 is a one-period stochastic variance, σ_t is a one-period stochastic volatility.

[2]In continuous time, the Feller condition $2F_{41} > H_{44}^2$ guarantees that the variance stays strictly positive. A formal modeling of this process in discrete time is achieved via a Poisson mixture of Gamma distributions (Gourieroux and Jasiak, 2006; Le, Singleton, and Dai, 2010). I use a direct discretization of the continuous-time square-root process to streamline the estimation of the model: I draw all parameters of the model together because the vector ε_{t+1} follows the multivariate normal distribution. I ensure that the variance remains positive by drawing it in logs.

[3]Price-dividend ratio and default premia are other variables used in consumption growth predictive regressions. See Colacito and Croce (2011) for details. I do not use these variables because connecting them to the pricing kernel requires an additional modeling effort. For example, the use of the price-dividend ratio must be accompanied by the model of the dividend growth process.

as a direct source of time variation in risk premia (Bansal and Shaliastovich, 2012; Drechsler and Yaron, 2011). Recently, Bansal, Kiku, Shaliastovich, and Yaron (2012) and Campbell, Giglio, Polk, and Turley (2012) have revisited the importance of the stochastic variance of consumption growth and emphasized its first-order implications for understanding the macro dynamics, as well as the time-series and cross-sectional properties of asset prices. In general, it is a challenging task to identify the stochastic variance in consumption data. My strategy of using a multi-variate system of consumption growth, inflation, and nominal yield to do so has a higher power because several variables have a stronger information content regarding common unobserved variance.[4]

The pricing kernel derived by applying preferences (2.1)-(2.2) to the consumption growth process (2.5) is

$$\log m_{t,t+1} \;\;=\;\; \log m + \eta' Y_t + q' \sigma_t \varepsilon_{t+1}, \tag{2.6}$$

where $\eta = (\eta_g, \ \eta_\pi, \ \eta_i, \ \eta_\sigma)'$ and $q = (q_g, \ q_\pi, \ q_i, \ q_\sigma)'$. The parameters of the vectors η and q are functions of the structural parameters of the model (Appendix A.1). Note that the pricing kernel naturally depends on all the states Y_t, but one of the states i_t^1 is a transformed asset price. Because the pricing kernel must value all assets in the economy, including the nominal bond, I impose a number of cross-equation restrictions on the VAR (2.5). As a result, the requirement of internal consistency of my model leads to a constrained vector autoregression. Except for these restrictions, I do not impose any other parameter constraints.

The equilibrium nominal yield is an affine function of all the model's states,

$$i_t^1 = A \log g_{t-1,t} + B \log \pi_{t-1,t} + C i_t^1 + D \sigma_t^2 + E,$$

where A, B, C, D, and E are the functions of the structural parameters describing the dynamics of the consumption growth and the preference parameters. I provide the full derivation of the equilibrium nominal yield in the Appendix A.1. To ensure that the

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[4]Carriero, Clark, and Marcellino (2012) document that a vector autoregression with common stochastic volatility factor efficiently summarizes the information content of several macroeconomic variables, such as GDP growth, consumption growth, growth of payroll employment, the unemployment rate, GDP inflation, the 10-year Treasury bond yield, the federal funds rate, and growth of business fixed investment. The authors justify this modeling approach using the observation that the pattern of estimated volatilities is often similar across variables.

price of the nominal bond is internally consistent, I require

$$A = B = D = E = 0, \tag{2.7}$$

$$C = 1. \tag{2.8}$$

The restrictions $A = B = E = 0$ and $C = 1$ are linear

$$\frac{G_{21}}{G_{11}} = \frac{G_{22}}{G_{12}} = \frac{G_{23} - 1}{G_{13}} = \frac{F_2 - \log \beta}{F_1} = \rho - 1, \tag{2.9}$$

whereas the restriction $D = 0$ is nonlinear

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\begin{aligned}
&\alpha(\alpha - \rho)(P + e_1)'HH'(P + e_1)/2 + e_2'Ge_4 - e_2'HH'e_2/2 \\
&-[(\alpha - \rho)P + e_1(\alpha - 1)]'HH'[(\alpha - \rho)P + e_1(\alpha - 1)]/2 \\
&+e_2'HH'[(\alpha - \rho)P + e_1(\alpha - 1)] - (\rho - 1)e_1'Ge_4 = 0, \tag{2.10}
\end{aligned}
$$

and depends on the endogenous parameters $P = (p_g, \ p_\pi, \ p_i, \ p_\sigma)'$ that show up in the solution of the value function

$$\log u_t = \log (U_t/c_t) = \log u + p_g \log g_{t-1,t} + p_\pi \log \pi_{t-1,t} + p_i i_t^1 + p_\sigma \sigma_t^2. \tag{2.11}$$

The parameters of the vector P are functions of the preference parameters and the parameters governing the dynamics of consumption growth. Vectors e_i that enter the non-linear restriction (2.10) are the corresponding coordinate vectors in a four-dimensional space. The nonlinear nature of the restriction (2.10), combined with the presence of the endogenous parameters, represents a serious challenge for the estimation. See Appendix A.1 for the model's solution and further details.

In summary, I model consumption growth via its joint dynamics with inflation and nominal interest rate. I allow for common stochastic variance and impose restrictions required for internal pricing consistency. This process for consumption growth, combined with recursive preferences, leads me to a fully articulated model of the pricing kernel.

2.3 Foreign exchange cash flow

To illustrate the basic risk-return relationship in the foreign exchange market, I consider the following investment strategy. At time t, the US representative investor buys a zero-coupon foreign bond of maturity τ and pays $\exp(-\tilde{i}_t^\tau)s_t/p_t$ US dollars (USD) in real

terms. At a future date $t+\tau$, the foreign bond pays back one unit of the foreign currency, i.e., $s_{t+\tau}/p_{t+\tau}$ USD in real terms. The excess τ- period log real return on this strategy,

$$\log rx_{t,t+\tau} = [\log s_{t+\tau} - \log s_t + \tilde{i}_t^\tau - i_t^\tau - \log \pi_{t,t+\tau}]/\tau,$$

is called a currency return because the currency price is a risky part of the strategy.[5] I use the following notation: exchange rate s_t is the price of one unit of foreign currency in terms of USD; i_t^τ ($\tilde{i}_t^\tau$) is the US (foreign) nominal yield of maturity τ; p_t is the US price index.

Next, I introduce the notion of the FX cash flow: $\delta_{t,t+\tau} = s_{t+\tau}/[s_t\pi_{t,t+\tau}]$. Strictly speaking, it is the real normalized cash flow of a foreign bond, or cash flow growth. I prefer to work with $\delta_{t,t+\tau}$ rather that with the original cash flow $s_{t+\tau}/p_{t+\tau}$ because the primer is stationary. The law of one price shows that the price of the foreign bond of maturity τ reflects the future currency risk at the horizon τ:

$$e^{-\tilde{i}_t^\tau} = E_t[m_{t,t+\tau}s_{t+\tau}/[s_t\pi_{t,t+\tau}]] = E_t[m_{t,t+\tau}\delta_{t,t+\tau}], \qquad (2.12)$$

where $m_{t,t+\tau}$ is the τ-period US (domestic) stochastic discount factor.

Since the study by Lustig and Verdelhan (2007), it is a standard practice in the literature to sort currencies into baskets depending on the level of the respective short interest rates (equivalently, on interest rate differentials) and to examine the covariance of the baskets' returns with some macroeconomic variables or return-based factors.[6] According to the law of one price (2.12), this sorting assigns currencies into baskets based on the price of the future currency exposure to risks at a fixed horizon τ. Equivalently, the existing literature has focused on understanding the nature of the risk-return relationship in the cross-section of currency baskets at a fixed horizon.

In contrast, I aim to understanding how the exposure of FX cash flows to the multiple sources of risk is priced at alternative horizons. Instead of sorting FX cash flows multiple times by the corresponding yields of maturity τ, I sort currencies into baskets based on the average yield in the corresponding foreign term-structures

$$\tilde{y}_t = \sum_{\tau=1}^{T} \tilde{i}_t^\tau.$$

[5]I assume that the investor holds foreign bond until maturity.
[6]Examples include, but not limited to, Burnside (2011), Burnside, Eichenbaum, Kleshchelski, and Rebelo (2011), Della Corte, Riddiough, and Sarno (2012), Lustig, Roussanov, and Verdelhan (2011), Lettau, Maggiori, and Weber (2012), Menkhoff, Sarno, Schmeling, and Schrimpf (2011), Mueller, Stathopoulos, and Vedolin (2012).

Thus, I build a cross-section of currencies with different exposure to the risks across multiple horizons. Other sorting strategies are possible. My view is that the average yield is a good proxy for the price of the multi-period exposure of FX cash flow to the risks. I perform robustness check and sort currencies into baskets based on the respective short interest rates as in the rest of the literature. My results remain similar.[7]

To characterize the risk-return relationship in the foreign exchange market at alternative horizons, I need the model of the joint dynamics of the pricing kernel and FX cash flows. Under the null of the model, I can perform analysis at any horizon without requiring more data at longer horizons, and, more importantly, I can deduce the role of every specific consumption risk for currency pricing. I augment the dynamics of the pricing kernel described in the previous section with the law of motion of the FX cash flow. To do so, I project the FX cash flow on the information set of the representative agent and the structural shocks and omit the orthogonal component:

$$\log \delta_{t,t+1} \quad = \quad \log \delta + \mu' Y_t + \xi' \sigma_t \varepsilon_{t+1} + \xi_v \sigma_t v_{t+1}, \tag{2.13}$$

where $\mu = (\mu_g, \ \mu_\pi, \ \mu_i, \ \mu_\sigma)'$, $\xi = (\xi_g, \ \xi_\pi, \ \xi_i, \ \xi_\sigma)'$ and v_{t+1} is an idiosyncratic shock. The omitted orthogonal component is irrelevant for the US pricing and does not affect statistical inference. The latter acts similarly to a linear regression, with an omitted variable that is orthogonal to regressors. By using the process (2.13), I make an additional assumption that world economies share the same volatility factor. Having a separate volatility factor for a foreign economy would be appealing; however, at the estimation stage the FX data at a quarterly frequency are not informative enough about it.

3 Data

3.1 Macro data

I use quarterly data on consumption growth, inflation, and three-month nominal yield from the second quarter of 1947 to the fourth quarter of 2011. In total, there are 259 observations. I collect consumption and price data from the NIPA tables of the Bureau of Economic Analysis. The nominal yield comes from CRSP. Appendix A.2 contains detailed data description.

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[7]The Online Appendix provides these results.

Table 1 provides basic descriptive statistics. The unconditional standard deviation of consumption growth is slightly higher than 1% annualized. This value is at least twice as low as the value over a longer time interval, including the Great Depression. Panels (a)-(c) of Figure 1 display the dynamics of these variables. It is clear from panels (b) and (c) that inflation and nominal rate tend to decrease during recessions. This observation is useful for the interpretation of empirical evidence later in the paper.

3.2   Currency and interest rate data

I collect daily data on twelve spot exchange rates from Thomson Reuters provided by Datastream. The sample contains the price of the Australian dollar, the British pound, the Canadian dollar, the Danish krone, the Euro, the Deutsche mark, the Japanese yen, the New Zealand dollar, the Norwegian krone, the South African rand, the Swedish krone, and the Swiss frank in terms of USD. The sample runs from the beginning of 1986 until the end of 2011. According to the latest report of the Bank of International Settlements, these currencies are among the twenty two currencies with the highest daily turnover, as of April 2010.

I use fixed income data from Datastream, Bloomberg and the dataset of Wright (2011). Wright (2011) provides detailed term-structure data for Australia, Canada, Germany, Japan, New Zealand, Norway, Sweden, Switzerland, and the UK until the first quarter of 2009. From the first quarter of 2009 until the last quarter of 2011, I compute swap implied interest rates for all of these countries but Germany. For Denmark, the Euro area, and South Africa, I compute the swap implied interest rates for the entire time interval. The term-structure data contain yields of forty maturities, from one quarter to ten years. Appendix A.2 (Table 14) describes data availability and sources of data for every country.

I choose currencies of developed countries that are used elsewhere in the literature.[8] Because of limited data availability on the term-structure of interest rates, my sample contains a smaller number of currencies. I work with quarterly currency quotes sampled at the end of the corresponding quarter. The choice of the data frequency corresponds to the frequency of consumption growth data.[9]

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[8]See Burnside, Eichenbaum, Kleshchelski, and Rebelo (2011), Lustig, Roussanov, and Verdelhan (2011), Lustig and Verdelhan (2007), Menkhoff, Sarno, Schmeling, and Schrimpf (2011), Rafferty (2011) among others.

[9]US consumption data are available at monthly, quarterly, and annual frequencies. It is well known that annual data are preferable but there are few observations to carry empirical work. I choose consumption data at a quarterly frequency as a compromise of quality and the number of available observations.

At the end of each quarter, I sort currencies into three baskets by the average yield in the foreign term-structure $\tilde{y}_t$. Because the number of currencies is small, I use only three portfolios. Basket "Low" contains the low interest rates currencies, basket "Intermediate" contains the intermediate interest rate currencies, and basket "High" contains the high interest rate currencies. Table 2 provides descriptive statistics of currency portfolio returns. The average return is monotonically increasing from basket "Low" to basket "High". Similar to Lustig and Verdelhan (2007), I find a spread in excess returns between basket "High" and basket "Low" of approximately 4.5% per year. Table 3 displays the dynamic composition of the baskets. Evidently, some currencies, e.g., the Japanese yen or the Swiss franc, remain in the same basket during the entire time period, so the basket re-balancing does not affect them. Other currencies, for example, the Canadian dollar or the Swedish krone, belong to each basket for several quarters.

4   Methodology

In this section, I describe my estimation approach. My ultimate goal is to estimate the joint dynamics of the stochastic discount factor and the FX cash flow. It suffices, however, to estimate the joint process for consumption growth and the FX cash flow. Recursive preferences applied to the dynamics of consumption growth pin down the dynamics of the stochastic discount factor. I assume that the idiosyncratic foreign exchange shock does not affect the dynamics of consumption growth; therefore, the estimation of the joint process is equivalent to a three-stage procedure: (1) estimation of the consumption growth process (2.5) with pricing restrictions (2.9) and (2.10), (2) identification of the structural shocks ε_{t+1} from the estimated reduced-form innovations in the vector autoregression, and (3) estimation of the cash flow process, taking into account identified structural shocks ε_{t+1}. My approach is free of the generated regressors' problem and provides the full distribution for all parameter estimates, stochastic variance, and structural shocks because I use the Bayesian methods. Below, I explain every stage in detail.

4.1 Estimation of the consumption growth process

I employ the Bayesian MCMC methods to estimate a vector autoregressive model of consumption growth,

$$Y_{t+1} \;\; = \;\; F + GY_t + \Sigma^{1/2}\sigma_t w_{t+1}, \tag{4.14}$$

with restrictions (2.9) and (2.10) and stochastic variance, where w_{t+1} are reduced-form innovations that are unknown linear functions of structural shocks: $H\varepsilon_{t+1} = \Sigma^{1/2}w_{t+1}$. Matrix $\Sigma^{1/2}$ is the Cholesky lower triangular matrix and vectors of shocks $w_{t+1} \sim \mathcal{N}(0, I)$ and $\varepsilon_{t+1} \sim \mathcal{N}(0, I)$ follow the multivariate normal distribution.

The key advantage of this estimation approach is that it allows to impose the required pricing restrictions (2.9) and (2.10) directly and delivers the estimated time-series of stochastic variance as a byproduct of the estimation routine. I carefully design the simulation method for the stochastic variance. In particular, I draw the log of variance; therefore, the variance itself never becomes negative or zero. My approach to estimating a vector autoregression with stochastic variance is different from standard methods used in applied macroeconomics.[10] In particular, I draw all the parameters of the vector autoregression jointly because the stochastic variance is a part of the vector of state variables.

The pricing consistency restrictions (2.9) and (2.10) are functions of the structural parameters governing the dynamics of consumption growth (2.5) and the preference parameters α, β, and ρ. Therefore, in addition to the twenty two parameters of the consumption growth process, there are three more preference parameters to estimate. This is a very challenging task, considering that the restriction (2.10) is nonlinear and requires a solution to the fixed point problem. I discuss the nature of the fixed point problem in Appendix A.3.

Parameters ρ and β appear in the linear restrictions (2.9) so that if I estimate F_1, F_2, G_{11}, G_{12}, G_{13}, and ρ (and this is a straightforward procedure), I can pin down $\log \beta$, G_{21}, G_{22}, and G_{23}. Instead, the parameter α enters only the nonlinear restriction (2.10). For this reason, it is not clear how to set up a prior for α and how to characterize its posterior distribution. Therefore, in this study I follow an easier yet still challenging route; that is, I assume the preference parameters and estimate the remaining twenty two parameters of the dynamics of consumption growth.[11] I account for the linear

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[10]See, for example, Cogley and Sargent (2005), Justiniano and Primiceri (2008), Primiceri (2005), Sargent and Surico (2010).

[11]I leave estimation of the preference parameters for the future research.

restrictions (2.9) by incorporating them directly in the parameter posterior distribution. I reject all the MCMC draws that violate the nonlinear restriction (2.10). To evaluate the nonlinear restriction, I solve the fixed-point problem at each draw; this process makes the estimation problem very time-consuming. If I had to estimate the preference parameters as well, in particular, α, the problem would be even more complicated.

I assume the following values for the preference parameters: $\alpha = -9$, $\beta = 0.9924$, and $\rho = 1/3$. The parameters α and ρ imply the preference for early resolution of uncertainty and have been extensively used in the literature to address a number of asset pricing puzzles. For example, by utilizing these preference parameters, Bansal and Yaron (2004) explain salient features of the equity market in an equilibrium framework of endowment economy; Hansen, Heaton, and Li (2008) empirically explain the value premium puzzle; whereas Bansal and Shaliastovich (2012) rationalize properties of the term-structure of nominal interest rates and the violation of the uncovered interest rate parity. In addition, in the international setting Colacito and Croce (2012) advocate EIS=3/2 ($\rho = 1/3$) as a value supported by empirical evidence gained through the lens of their structural model. Finally, I borrow the value of the subjective discount factor β from Hansen, Heaton, and Li (2008).

4.2 Identification

To recover structural shocks ε_{t+1} from the reduced-form innovations w_{t+1}, I augment the model with a number of economically motivated identifying restrictions as is usually done in structural vector autoregressions in applied macroeconomics.[12] The natural question is the number and the type of restrictions that should be imposed. Rothenberg (1971) provides a necessary condition, also known as the order condition, which says that to identify a system of n equations there must be $n(n-1)/2$ zero restrictions imposed. I have a system of four equations. Therefore, to identify the structural shocks ε_{t+1}, it is necessary to impose six restrictions. Theorem 1 in Rubio-Ramirez, Waggoner, and Zha (2010) provides a sufficient condition, also known as the rank condition, stating that the location of restrictions in the matrix H matters.

I choose to work with zero restrictions. The necessary and sufficient conditions together with several additional considerations guide me towards particular identification schemes. Firstly, the stochastic variance σ_t^2 follows the square root process,

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[12]See Blanchard and Quah (1989), Cochrane (1994), Christiano, Eichenbaum, and Evans (1999), Eichenbaum and Evans (1995), Leeper, Sims, Zha, Hall, and Bernanke (1996), Stock and Watson (2012), Uhlig (2005) among others.

meaning that three restrictions on matrix H have been imposed from the beginning: $H_{41} = H_{42} = H_{43} = 0$. Next, economically, there must be no zero restrictions on the elements of the third row of the matrix H. The third variable in the system is the nominal rate. It is an equilibrium outcome, and hence, an affine function of the model's states. In principle, the nominal rate might not load materially on one state or another, but a priori, it would be unreasonable to restrict the model in any possible way. Finally, two equations and three more restrictions remain. Here, I follow Theorem 1 from Rubio-Ramirez, Waggoner, and Zha (2010) and find that the only two combinations of three zero restrictions (1) $H_{12} = H_{13} = H_{23} = 0$ and (2) $H_{13} = H_{21} = H_{23} = 0$ guarantee that the model is globally identified.

Identification $H_{12} = H_{13} = H_{23} = 0$ is labeled "Fast Inflation" because inflation reacts contemporaneously to a short-run consumption shock, whereas consumption growth reacts to a current inflation shock with a one-quarter delay. Table 4 displays the corresponding location of zero restrictions. Identification $H_{13} = H_{21} = H_{23} = 0$ is labeled "Fast Consumption" because consumption reacts contemporaneously to an inflation shock, whereas inflation reacts to a short-run consumption growth shock with a one-quarter delay. Table 5 displays the corresponding location of zero restrictions. I borrow the terminology from structural VARs in applied macroeconomics.

I name the direct consumption shock $\varepsilon_{g,t+1}$ the short-run shock and the shock $\varepsilon_{i,t+1}$ the long-run risk shock, based on their estimated properties. In particular, the impact of the shock $\varepsilon_{g,t+1}$ on consumption growth is concentrated in the short-run, whereas the cumulative impact of the shock $\varepsilon_{i,t+1}$ on consumption growth dominates at long horizons.

The identification of the long-run risk shock $\varepsilon_{i,t+1}$ and the variance shock $\varepsilon_{\sigma,t+1}$ is exactly the same in both identification schemes. The shock $\varepsilon_{i,t+1}$ is identified in the spirit of Bansal and Yaron (2004), i.e., the long-run risk shock affects expected consumption growth but not consumption growth itself, and does not feed into the variance process. The identification of the variance shock $\varepsilon_{\sigma,t+1}$ has a flavor of the structural assumptions of Colacito (2009) who allows for non-zero conditional correlations between consumption growth and stochastic variance and expected consumption growth and stochastic variance. Identification of the short-run consumption shock $\varepsilon_{g,t+1}$ and the inflation shock $\varepsilon_{\pi,t+1}$ is different across the identification schemes, as discussed above.

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4.3 Estimation of the FX cash flow process

The estimation of the FX cash flow process (2.13) becomes straightforward once the structural shocks ε_{t+1} from the VAR (2.5) are identified. Intuitively, the cash flow process is a part of the vector autoregression that also includes the dynamics of consumption growth, inflation, nominal yield, and stochastic variance. The FX cash flow does not Granger-cause the economic states, whereas the economic states do cause the FX cash flow. In other words, there is nothing new to learn from the dynamics of foreign exchange cash flow that is not already contained in the dynamics of economic states. Given this property, the estimation of the joint distribution of the economic states and foreign exchange cash flow can be performed in two steps, as follows: (1) estimate the model of consumption growth (2.5) and (2) use the results from (1) to estimate the foreign exchange cash flow, i.e., measure the loadings of the corresponding cash flow on economic states and structural shocks. Because a two-stage estimation is equivalent to the estimation of the joint process, the problem of generated regressors does not arise.

Effectively, estimating the FX cash flow process is almost identical to running a linear regression because the full distribution of the stochastic variance σ_t^2 and structural shocks ε_{t+1} are already known, as a byproduct of the Bayesian MCMC approach. Components such as $\sigma_t \varepsilon_{g,t+1}$ in the process (2.13) act as additional regressors to the economic states. I use the Bayesian MCMC methods to estimate the FX cash flow process. I provide the details of the estimation algorithm and discuss my choice of priors in the Online Appendix.

4.4 Shock elasticity

In this section, I describe how I quantify prices and quantities of consumption risks in the cross-section of currency baskets at alternative horizons. I follow the idea of dynamic value decomposition of Hansen (2012) and, in particular, I use shock-exposure and shock-price elasticities of Borovička and Hansen (2011) and Borovička, Hansen, Hendricks, and Scheinkman (2011). Shock-exposure elasticity and shock-price elasticity are marginal metrics of quantity and price of risk, respectively.

The importance of a distinct source of risk for a cash flow is measured by the magnitude of the risk premium earned because of the cash flow's exposure to the risk. Two metrics matter: quantity of risk (exposure) and price of risk (compensation per unit of exposure). In a dynamic world with multiple sources of risk, the total risk premium

is a compensation for cash flow's exposure to all the sources of risk at many horizons. Thus, to shed light on the relative importance of one source of risk, it is necessary to isolate one shock of that type and study its pricing implications for cash flow $\delta_{t,t+\tau}$ across different horizons τ. In this case, the quantity and price of risk depend on the time gap $\tau - 1$ between the moment when the shock is realized and the moment when the shock impacts the cash flow. This dependence on time creates a term-structure of risks and their prices.

Borovička and Hansen (2011) describe in detail how to characterize the term-structure of risks and their prices in a structural model with stochastic variance in discrete time. I illustrate their approach in a simple example by examining the role of the variance risk. Appendix A.4 provides the formal derivation of shock elasticities in the context of my model.

To characterize the role of the variance risk ε_σ, Borovička and Hansen (2011) propose to undertake the following steps. First, they change the exposure of the cash flow $\log \delta_{t,t+\tau}$ to the risk $\sigma_t \varepsilon_{\sigma,t+1}$.[13] To do so, they introduce a perturbation

$$\log h(\mathrm{v}) = \gamma(\mathrm{v}, \sigma_t) + \mathrm{v}\sigma_t \varepsilon_{\sigma,t+1},$$

where the functional form of $\gamma(\mathrm{v}, \sigma_t)$ is not important and v is a scalar, and add this perturbation to the original multi-period cash flow $\delta_{t,t+\tau}$:

$$\log \bar{\delta}_{t,t+\tau} = \log \delta_{t,t+\tau} + \log h(\mathrm{v}).$$

As a result, they change the amount of the variance risk in the cash flow by the value v at time $t+1$. Next, the authors study how the log of the expected cash flow changes in response to a change in the amount of risk, when the change is marginal, i.e., they compute the following derivative

$$\ell_\delta(Y_t, \tau) = \left. \frac{\mathrm{d} \log E_t[\delta_{t,t+\tau}]}{\mathrm{d} \log h(\mathrm{v})} \right|_{\mathrm{v}=0} = \left. \frac{\mathrm{d} \log E_t[\bar{\delta}_{t,t+\tau}]}{\mathrm{d}\mathrm{v}} \right|_{\mathrm{v}=0} \tag{4.15}$$

and call the result the shock-exposure elasticity. Similarly, they study how the log risk premium changes in response to a change in the amount of risk, when the change is marginal, i.e., they compute the following derivative

$$\ell_p(Y_t, \tau) = \left. \frac{\mathrm{d} \log E_t[rx_{t,t+\tau}]}{\mathrm{d} \log h(\mathrm{v})} \right|_{\mathrm{v}=0} = \left. \frac{\mathrm{d} \log E_t[\bar{\delta}_{t,t+\tau}]}{\mathrm{d}\mathrm{v}} \right|_{\mathrm{v}=0} - \left. \frac{\mathrm{d} \log E_t[\bar{\delta}_{t,t+\tau} m_{t,t+\tau}]}{\mathrm{d}\mathrm{v}} \right|_{\mathrm{v}=0} \tag{4.16}$$

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[13]Without loss of generality, assume that $\sigma_t \varepsilon_{\sigma,t+1}$ has a unit standard deviation.

and call this object the shock-price elasticity. The derivative with respect to $\log h(\mathrm{v})$ is effectively a derivative with respect to the random variable $\mathrm{v}\sigma_t\varepsilon_{\sigma,t+1}$. In continuous time, such a derivative is known as the Malliavin derivative. Borovička, Hansen, Hendricks, and Scheinkman (2011) show that it is equal to the directional derivative in the right-hand side of (4.15) or (4.16) in continuous time. Borovička and Hansen (2011) simply adopt the directional derivative as a definition of the shock elasticity in discrete time.

The shock-exposure elasticity $\ell_\delta(Y_t, \tau)$ is marginal quantity of risk, whereas the shock-prices elasticity $\ell_p(Y_t, \tau)$ is marginal price of risk, or the marginal Sharpe ratio. The elasticities depend on the time elapsed since the shock has been realized until it impacts the cash flow and on the information set Y_t. These marginal metrics can be viewed as asset pricing counterparts to cumulative impulse response functions. Shock elasticities are specifically designed to study asset pricing implications of structural models with stochastic variance (or other types of nonlinearity).

In a linear model, marginal metrics of quantity and price of risk correspond to their average counterparts. Therefore, the shock-exposure elasticity is the cumulative impulse response function of the multi-period log cash flow (multi-period quantity of risk), whereas the shock-price elasticity is the cumulative impulse response function of the negative of the multi-period log stochastic discount factor (average multi-period Sharpe ratio). Section 2.2 of Borovička and Hansen (2011) illustrates this equivalence.[14] However, in a model with stochastic variance (or other types of nonlinearities), shock elasticities do not coincide with cumulative impulse response functions, and have a different interpretation. This difference is critical because only shock elasticities can describe risks in isolations in the presence of nonlinearities.

My model contains three types of risk, namely, the short-run consumption risk, the inflation risk, and the long-run consumption risk, which enter the model linearly. Therefore, the shock-exposure and shock-price elasticity for these risks have a standard interpretation of average quantity and price of risk. Shock elasticities for the variance risk has interpretation of marginal quantity and price of risk. In this case, prices of risk associated with different currency baskets are different, i.e., they are basket specific. This is a direct manifestation of nonlinearity.

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[14]Intuitively, the equivalence holds because the nonlinearities, for which shock elasticities additionally account, are absent. Roughly speaking, the cumulative impulse response function requires computing the expectation of the log, whereas the shock elasticity requires computing the opposite, i.e., the log of the expectation. In a linear model the order does not matter.

5   Results

I present my findings in the following order. I start with a discussion of the estimated dynamics of the structural VAR. Next, I analyze how foreign exchange cash flows are sensitive to the four identified sources of consumption risk at alternative horizons. Finally, I examine how these risk exposures are priced at alternative horizons.

5.1   Macro dynamics

I use the data displayed in panels (a)-(c) of Figure 1 to estimate the model (4.14) with the consistency restrictions (2.9) and (2.10). The Online Appendix summarizes the diagnostics of fitting errors based on which I conclude that the model has a good fit. One of the outputs of the estimation procedure is the estimated path of the unobservable stochastic variance σ_t^2, another output of the estimation procedure is the expected consumption growth $E_t \log g_{t,t+1}$ displayed in panel (a) of Figure 1.

I take the square root of σ_t^2 and scale it appropriately, so that the series represents the stochastic volatility of consumption growth. I display this series in panel (d) of Figure 1. The annualized volatility of consumption growth varies from 0.6% to 2.12%. It captures the important economic periods: the volatility is high after the Second World War, during the oil crises, the monetary experiment, and the recent financial crisis, and volatility is relatively low during the Great Moderation.

Table 6 reports the parameter estimates for the elements of the matrices F, G, and Σ. The element G_{44} is of special interest because it characterizes the persistence of the stochastic variance. The estimated half-life of the variance component is $\log 2/(1 - G_{44}) = 13$ quarters. It is particularly interesting to compare the estimate of G_{44} with the corresponding values used in calibrations elsewhere in the literature. Similar to the specification of the consumption growth process in Bansal and Yaron (2004), my model has only one stochastic variance factor.[15] I proceed by comparing the estimate of G_{44} with the corresponding parameter values used in different calibrations of the Bansal and Yaron (2004) model, e.g., in Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012a) and Bansal, Kiku, and Yaron (2012b). These values are 0.9615, 0.9949, and 0.997 on a quarterly basis, respectively; they are higher than my point estimate of G_{44} which is

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[15]In contrast to the theoretical long-run risk literature, I specify the stochastic variance not as an autoregressive process but as a discretized version of the square-root process.

0.9476. However, the persistence parameter used by Bansal and Yaron (2004) is within the confidence interval of the estimated parameter G_{44}.

The estimated persistence of the expected consumption growth is 0.81 with the 95% confidence interval from 0.71 to 0.90.[16] These magnitudes are somewhat smaller than the values used in standard calibrations of the long-run risk models.[17] The expected consumption growth loads significantly on all the observables used in the estimation with the largest in absolute terms loading on the nominal yield ($G_{13} = 0.38$).[18] Because of the dominant role of the nominal yield, the cyclical properties of the expected growth and the nominal yield are similar. Occasionally, however, the expected consumption growth mirrors the dynamics of other variables. For example, during the recent financial crisis the dynamics of the expected consumption growth is mostly related to the dynamics of inflation with a negative sign, whereas during the economic downturn of 1958 the expected consumption growth closely tracks the evolution of the realized consumption growth.

Table 7 contains the estimates for the parameters of the matrix H. Under both identification schemes, I find that a positive variance shock leads to a positive contemporaneous move in inflation, whereas a positive short-run consumption shock leads to a positive contemporaneous move in the nominal yield. Additionally, under "Fast Inflation" a positive short-run consumption shock leads to an increase in inflation, whereas under "Fast Consumption", a positive inflation shock increases consumption growth. This impact of the structural shocks on the states of the model affects the one-period prices of risks attached to them.

5.2   Term-structure of exposures of FX cash flows to the multiple sources of consumption risk

Table 8 and Table 9 describe the distribution of the parameters of the cash flow process estimated for all currency baskets under both identification schemes. For the one-period exposures, the parameters ξ_g, ξ_π, ξ_i, and ξ_σ are of central interest. These parameters

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[16]I compute the persistence parameter as an autocorrelation of the expected consumption growth $corr(E_t \log g_{t,t+1}, E_{t-1} \log g_{t-1,t})$.

[17]For example, Bansal and Yaron (2004) use the autoregressive parameter of 0.94, whereas Bansal, Kiku, and Yaron (2012a) use the value of 0.93. I refer to the parameter values corresponding to the consumption dynamics at a quarterly frequency.

[18]The loading of the expected consumption growth on realized consumption growth is $G_{11} = 0.2$, the loading on inflation is $G_{12} = -0.2$, and the loading on consumption variance is $G_{13}/\Sigma_{11} = 5.7$. Note that the consumption variance is several orders lower than consumption growth or inflation.

are the loadings on the vector of structural shocks $\sigma_t \varepsilon_{t+1}$ in the cash flow process, and, therefore, can be interpreted as the quantity of the short-run risk, inflation risk, long-run risk, and variance risk, respectively. Under both identification schemes, the cash flow of basket "Low" loads negatively on the long-run risk shock, the cash flow of basket "Intermediate" loads positively on the short-run consumption shock and inflation shock and negatively on the long-run risk shock, and the cash flow of basket "High" loads positively on the short-run consumption shock, inflation shock, and long-run risk shock. Thus, at horizon of one quarter, the cash flows of basket "Low" and basket "Intermediate" serve as hedges against the long-run risk shock; in other words, cash flows increase after a negative long-run risk shock.

For multi-period horizons, the parameters μ_g, μ_π, μ_i, and μ_σ become important. In conjunction with the parameters of the matrices G and H, they determine how shocks propagate across time in the cross-section of FX cash flows. Under both identification schemes, cash flows are predictable. Consumption growth, inflation and stochastic variance have forecasting power for basket "Low"; inflation and nominal rate have forecasting power for basket "Intermediate"; and consumption growth, inflation and nominal rate have forecasting power for basket "High".

In many cases, the contemporaneous and future effects of the same shock are opposite. For example, a positive short-run consumption shock $\varepsilon_{g,t+1}$ contemporaneously decreases the cash flow of basket "Low" ($\xi_g < 0$) but increases the corresponding future one-period cash flow ($\mu_g > 0$). Therefore, it is hard to gauge whether the cumulative effect of $\varepsilon_{g,t+1}$ on the multi-period cash flow of basket "Low" is positive or negative on the basis of the estimated parameters alone. Shock-exposure elasticities are helpful in this regard.

Figure 2 and Figure 3 display the shock-exposure elasticity under the "Fast Inflation" identification and the "Fast Consumption" identification, respectively. To plot the graphs, I set the stochastic variance σ_t^2 to be equal to 1, i.e., to its long-run mean.[19] Shock-exposure elasticities for short-run consumption shock, inflation shock, and long-run risk shock can be interpreted as quantities of risk in a standard sense (for example, $\xi_g \sigma_t$ is a one-period quantity of the short-run risk associated with some FX cash flow). These shocks do not feed into the stochastic variance process; therefore, the average metrics of price and quantity of risk coincide with their marginal counterparts. In contrast, shock exposure elasticity for the variance shock has an interpretation of the

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[19]The shock-exposure elasticities scale up and down depending on the magnitude of the stochastic variance.

marginal quantity of risk: marginal change in the expected cash flow due to a marginal change in the volatility of the underlying shock.

To highlight the difference between average and marginal quantity of risk, I interpret a currency with a negative exposure elasticity to the short-run consumption shock, inflation shock, or long-run risk shock as an average hedge against the corresponding shock, whereas a currency with a negative cash flow exposure elasticity to the variance shock as a marginal hedge against the shock. Bearing this in mind, I proceed by looking at the cross-sectional implications of the exposure elasticities.

Under both identification schemes, there is significant cross-sectional heterogeneity of the currency exposure elasticities to the long-run risk shock and inflation shock across all horizons from one quarter to ten years. The sensitivity to the long-run risk shock is lowest for basket "Low" and highest for basket "High", whereas the sensitivity to the inflation shock is lowest for basket "Intermediate" and highest for basket "High". Differences in the exposure elasticities of basket "Low" and basket "Intermediate" to the long-run risk and inflation shocks are not statistically significant for multi-period horizons, although the differences are economically meaningful in case of the elasticity exposure to the long-run risk shock. Pair-wise differences in the exposure elasticities of basket "Low" and basket "High" and basket "Intermediate" and basket "High" to the long-run risk and inflation shocks are economically and statistically significant at all horizons.[20] Thus, the low and intermediate interest rate currencies are average hedges against the long-run and inflation risks.

The loadings of FX cash flows on the variance risk are not significantly different in the cross-section. The variance risk matters in a different respect. Under both identification schemes at horizons longer than three years, FX cash flows are the most sensitive to the variance shock. To gauge the cumulative impact of the variance shock on the cash flows, it is helpful to consider an example. A sensitivity of the high interest rate currencies to the variance risk at horizon of ten years is -0.034 (a sensitivity to the long-run risk is 0.026, to the inflation shock – 0.015, to the short-run consumption shock – 0.02), whereas the corresponding metric on a one-period horizon is $\xi_\sigma = 0.005$. The variance shock has a long-lasting impact on FX cash flows. At long horizons, all currency baskets are marginal hedges against the variance shock; that is, FX cash flow marginally decreases as a result of a marginal increase in the exposure to the variance shock.

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[20]Under the "Fast Consumption" identification, the differences in the exposure elasticities of basket "Low" and basket "High" to the inflation risk are significant only at horizons from three quarters to five years. To avoid overcrowding the figures, I do not display the confidence bounds for shocks elasticities. Results are available upon request.

5.3 Term-structure of prices of multiple sources of consumption risk

In the previous section, I have documented the following findings: (1) there are economically and statistically significant differences in exposures of currencies to the inflation and long-run risks at multiple horizons and to the short-run consumption risk at a one-period horizon only and (2) the sensitivity of currency baskets to the variance shock is large in absolute value and negative at long horizons without cross-sectional differences among the baskets. The next natural question concerns how the currency exposure to the consumption risks is priced at different horizons. Namely, it is important to understand if the cross-sectional differences in quantity of short-run risk, long-run risk, or inflation risk across the currency baskets lead to a material difference in risk premia in the cross-section at different horizons.

I start characterizing the prices of risks from a one-period perspective. Table 10 describes the distribution of p_g, p_π, p_i, and p_σ that are the parameters of the value function (2.11). Parameters p_g and p_i are positive and statistically significant, whereas the confidence intervals for the parameters p_π and p_σ include zero. Standard calibrations of the long-run risk models (see, for example, Bansal and Yaron, 2004; Bansal, Kiku, and Yaron, 2012a; Drechsler and Yaron, 2011) produce a negative value for p_σ and a positive loading on the $\sigma_t \varepsilon_{\sigma,t+1}$ in the stochastic discount factor (negative price of the variance risk).

At this stage, it is important to make three remarks. First, the preference parameters and the parameters p_g, p_π, p_i, and p_σ are not the only determinants of the signs of the prices of risk. The negative of the vector of the one-period prices of risks, q, depends on H: $q = H'[(\alpha - \rho)P + e_1(\alpha - 1)]$ (see Appendix A.1). The matrix H is not diagonal, and therefore, the interaction between H and P matters. Second, in my model σ_t^2 can play the following two roles: (1) the variance factor and (2) the predictability factor of the future consumption growth (similar to Backus, Routledge, and Zin, 2010). Higher variance today could be associated with higher expected consumption growth in the future, so in general, the sign of p_σ is undetermined on the basis of economic intuition alone. Finally, as the Online Appendix shows for the Bansal and Yaron (2004) model, it could be difficult to precisely identify the parameter p_σ from the data.

Table 11 describes the distribution of q_g, q_π, q_i, and q_σ (elements of the vector q) under both identification schemes. The absolute value of the price of the short-run consumption shock is higher under the "Fast Inflation" identification. The inflation shock carries a statistically significant price of risk only under the "Fast Consumption" identification. The distribution of q_i and q_σ is identical across the schemes because

these risks are identified in exactly the same manner. The long-run risk shock carries a statistically significant positive price of risk $(-q_\sigma)$. The confidence interval for the price of the variance risk includes zero. High uncertainty about p_σ leads to a high uncertainty about the price of the variance risk.

I take into account the properties of the prices of risks and one-period exposures of FX cash flows to the risks and analyze the model implications for the cross-section of one-period risk premia in Table 12 and Table 13. Basket "Low" is associated with a negative risk premium (approximately -2% annualized) because it pays well in bad states of the world when a negative long-run risk shock is realized. The average historical return on basket "Low" is -2.52% which falls within the confidence interval of the one-period total real risk premium attached to this basket. Basket "High" is associated with a positive risk premium (approximately 3.2% annualized) because its cash flow is positively exposed to all the risks which carry positive price. The average historical return on basket "High" is 2%, and is within the confidence interval of the one-period total real risk premium attached to this basket. Finally, basket "Intermediate" earns a negative risk premium of approximately -0.35% annualized because its cash flow is more sensitive to the long-run risk shock, and this sensitivity is negative. Similar to the other baskets, the historical average return on basket "Intermediate" (-0.71%) is within the model implied 95% confidence interval of the one-period total real risk premium.

To summarize, the level and the spread of the excess returns in the cross-section of currencies is fully explained by the exposure of currencies to the priced sources of consumption risk. Under the "Fast Inflation" identification, exposure to the long-run risk shock accounts for 52% of the one-period spread of excess returns between the high and low interest rate currencies, whereas the remaining 48% are due to the different exposure of the currency baskets to the short-run consumption shock. Under the "Fast Consumption" identification, exposures to the long-run risk shock, the short-run consumption shock, and the inflation shock contribute 48%, 26%, and 26%, respectively, to the spread of real excess returns.

I analyze the multi-period prices of risks by examining the shock price elasticities displayed in Figure 4 and Figure 5.[21] The price elasticity of the short-run consumption shock, the inflation shock and the long-run risk shock corresponds to the negative of the cumulative impulse response function of the multi-period log stochastic discount. This works similarly to a linear model without stochastic variance because these shocks do not feed into the process for stochastic variance. Therefore, the marginal price of risk

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[21]As in the case of exposure elasticity, I plot price elasticity by setting $\sigma_t^2 = 1$.

associated with these shocks is also the average price of risk, or average Sharpe ratio for log returns.

Such an interpretation is not appropriate for the price elasticity for the variance shock. The variance shock feeds into the variance process, and therefore, it is associated with important nonlinearities in the model. The price elasticity of the variance shock is a marginal change in the risk premium caused by the marginal change in the exposure to the source of risk, i.e., a marginal Sharpe ratio for log returns. The price elasticity for the variance shock is cash flow dependent because its marginal price of risk is not equal to its average price of risk.

To put the magnitudes displayed in Figure 4 and Figure 5 into perspective, I refer to a number of studies that report Sharpe ratios for different currency strategies. Table 3 in Ang and Chen (2010) reports an annualized Sharpe ratio of 0.64 for a currency portfolio based on the level of the yield curve and 0.81 for a currency portfolio based on the slope of the yield curve; Table 1 in Burnside (2011) reports an annualized Sharpe ratio of 0.90 for the equally-weighted carry trade and 0.63 for the HML carry trade; Table 1 in Lustig, Roussanov, and Verdelhan (2012) documents an annualized Sharpe ratio of 0.66 for the dollar carry trade.[22]

These numbers are not exact counterparts to the prices of risk that I document in the paper. In particular, I report Sharpe ratios for log returns, consider different strategies, and use different data. However, I believe these numbers are still informative and could be used as a rough benchmark. The one period log Sharpe ratios for the short-run consumption shock and long-run risk shock are smaller than their multi-period counterparts but already substantial enough against the numbers quoted for currency strategies elsewhere in the literature (see above). For example, the annualized Sharpe ratio due to the short-run consumption shock is approximately 0.52 or 0.58 (depending on the identification strategy) and due to the predictability shock is 0.66.

The risk premium of all currency baskets at all investment horizons is especially sensitive to the long-run risk under both identification schemes. This finding, in conjunction with the substantial spread in quantity of the long-run risk across currency baskets, is the

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[22]Ang and Chen (2010) describe a currency strategy based on the level (slope) of the yield curve as one that entails going long in a currency with a high level factor (low term spread) and short in a currency with a low level factor (high term spread); Burnside (2011) defines the equally weighted carry trade as the average of up to twenty individual currency carry trades against the US dollar; Lustig, Roussanov, and Verdelhan (2012) determine dollar carry trade as a strategy of going long in all available one-month currency forward contracts when the average forward discount of developed countries is positive and short otherwise.

main result of the paper. Currency baskets carry significantly different compensation for the long-run risk at all horizons from one quarter to ten years. The spread in compensation decreases with the investment horizon: basket "Low" acts as a weaker hedge, whereas basket "Intermediate" loses its hedging capability completely. Nonetheless, the spread between the corner baskets remains statistically significant.

The price of inflation risk is statistically significant at all horizons under the "Fast Consumption" identification only. In this case, the significant difference in exposure to inflation risk between basket "Low" and basket "High" (basket "Intermediate" and basket "High") leads to a significant spread of excess returns at all horizons (at horizons shorter than five years). The cross-sectional spread of the inflation risk premia is smaller than the cross-sectional spread of the long-run risk premia because the price of the long-run risk is more than double that of the inflation risk.

Finally, the sensitivity of the currency risk premia to the variance risk is relatively small at all investment horizons. This finding demonstrates that a high sensitivity of a cash flow to a specific source of risk does not necessarily lead to a high risk compensation. Moreover, the positive marginal price of the variance risk, suggests that all currency baskets act as a marginal hedge against the unfavorable variance shock.

6 Conclusion

In this paper, I provide novel evidence of how multiple sources of consumption risk are priced in the foreign exchange market at short and medium horizons, from one quarter to ten years. I accomplish the task by examining the role of the consumption risks through the lens of the vector autoregressive process of the joint dynamics of consumption growth, inflation, and a three-month nominal yield with stochastic variance and structural restrictions derived under recursive preferences.

I establish four structural consumption shocks, including the short-run consumption risk, the inflation risk, the long-run consumption risk, and the variance risk. I find that the compensation for currency exposure to these risks at the horizon of one quarter matches both the level and the cross-sectional spread of currency risk premia. I document the prominent role of the long-run consumption risk: (1) it carries the highest price of risk (annualized average log Sharpe ratio is 0.66 at the horizon of one quarter and higher at longer horizons), and (2) it contributes the most to the level and to the spread of excess returns between baskets of high and low interest rate currencies at short

and medium horizons (at the horizon of one quarter, this risk explains at least 42% of the spread).

The role of other sources of risk is limited. The short-run consumption risk is priced in the cross-section of currency returns at the horizon of one quarter only, where it explains at least 26% of the corresponding spread of excess returns between high and low interest rate currencies. The inflation risk matters at multiple horizons if consumption growth is a faster variable than inflation (consumption growth reacts to the inflation shock within a quarter whereas inflation reacts to the short-run consumption shock with a delay of one quarter). This risk explains a lower fraction of the spread in excess returns in comparison with the long-run risk because its price of risk is less than a half of that for the long-run risk (annualized average log Sharpe ratio of the inflation risk at the horizon of one quarter is 0.26). Finally, I find that all currency baskets are uniformly highly sensitive to the variance risk at horizons longer than three years, although the compensation for this exposure is small.

I leave at least two interesting avenues for the future research. The first question is the estimation of the preference parameters, perhaps starting with the parameter of the elasticity of intertemporal substitution and the subjective discount factor. The second direction of research is further exploration of the role of the variance risk in macroeconomy and asset markets by utilizing assets that are informative about this type of risk at the estimation stage.

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Table 1
Properties of macro economic variables

	Mean	Std Dev	Skewness	Kurtosis	N observations
$\log g_{t,t+1}$	0.0048	0.0052	-0.45	4.04	259
$\log \pi_{t,t+1}$	0.0083	0.0076	0.81	5.30	259
i_t^1	0.0113	0.0076	0.93	4.13	259

Notes. Descriptive statistics for consumption growth, inflation, and nominal yield. Sample period: second quarter of 1947 – fourth quarter of 2011. Quarterly.

Table 2
Properties of real log excess returns

	Mean	Std Dev	Skewness	Kurtosis	Autocorrelation
Basket "Low"	-0.0063	0.0517	0.38	3.07	0.01
Basket "Intermediate"	-0.0018	0.0432	0.09	3.81	0.15
Basket "High"	0.0050	0.0502	0.03	3.62	0.12

Notes. The three currency baskets are formed by sorting currencies by their corresponding average yields at a quarterly basis. Average yields are computed for each currency's term-structure at each point of time. Sample period: 1986 – 2011. Quarterly.

Table 3
Composition of currency baskets

Currency	Basket "Low"	Basket "Intermediate"	Basket "High"
Australia	0	23	76
Canada	20	75	8
Denmark	11	70	12
Germany	34	16	2
Euro area	17	12	0
Japan	103	0	0
Norway	1	24	30
New Zealand	4	10	73
Sweden	32	29	15
Switzerland	95	0	0
UK	5	50	48
South Africa	0	0	58

Notes. Table entry shows the number of periods each currency belongs to each basket. Sample period: 1986 – 2011, at a quarterly frequency.

Table 4
Identification "Fast Inflation"

	$\varepsilon_{g,t+1}$	$\varepsilon_{\pi,t+1}$	$\varepsilon_{i,t+1}$	$\varepsilon_{\sigma,t+1}$
Consumption eq	H_{11}	0	0	H_{14}
Inflation eq	H_{21}	H_{22}	0	H_{24}
Interest rate eq	H_{31}	H_{32}	H_{33}	H_{34}
Variance eq	0	0	0	H_{44}

Notes. A globally identified system. Inflation reacts to a consumption shock ε_g contemporaneously, whereas consumption growth reacts to an inflation shock ε_π with a delay of one period.

Table 5
Identification "Fast Consumption"

	$\varepsilon_{g,t+1}$	$\varepsilon_{\pi,t+1}$	$\varepsilon_{i,t+1}$	$\varepsilon_{\sigma,t+1}$
Consumption eq	H_{11}	H_{12}	0	H_{14}
Inflation eq	0	H_{22}	0	H_{24}
Interest rate eq	H_{31}	H_{32}	H_{33}	H_{34}
Variance eq	0	0	0	H_{44}

Notes. A globally identified system. Consumption growth reacts to an inflation shock ε_π contemporaneously, whereas inflation reacts to a consumption shock ε_g with a delay of one period.

Table 6
The model of consumption growth. Parameter estimates

Parameter	Estimate	Confidence interval, 95%
F_1	-0.0004	(-0.0019, 0.0011)
F_2	-0.0074	(-0.0008, 0.0013)
F_3	-0.0002	(-0.0006, 0.0002)
F_4	0.0525	(0.0229, 0.0843)
G_{11}	0.2009	(0.1051, 0.3029)
G_{12}	-0.2000	(-0.2827, -0.1204)
G_{13}	0.3792	(0.2896, 0.4654)
G_{14}	0.0017	(0.0009, 0.0026)
G_{21}	-0.1339	(-0.2019, -0.0701)
G_{22}	0.1333	(0.0803, 0.1885)
G_{23}	0.7472	(0.6897, 0.8069)
G_{24}	0.0046	(0.0036, 0.0057)
G_{31}	0.0721	(0.0378, 0.1060)
G_{32}	0.0183	(-0.0061, 0.0422)
G_{33}	0.9635	(0.9425, 0.9845)
G_{34}	0.0003	(3.66e-5, 0.0006)
G_{44}	0.9476	(0.9156, 0.9771)
Σ_{11}	3.10e-5	(2.15e-5, 4.53e-5)
Σ_{12}	8.85e-6	(3.06e-6, 1.60e-5)
Σ_{13}	2.68e-6	(1.07e-6, 5.05e-6)
Σ_{14}	-0.0002	(-0.0004, 2.51e-5)
Σ_{22}	3.87e-5	(2.80e-5, 5.35e-6)
Σ_{23}	2.90e-6	(9.22e-7, 5.85e-6)
Σ_{24}	0.0003	(4.80e-5, 0.0005)
Σ_{33}	2.71e-6	(1.92e-6, 3.87e-6)
Σ_{32}	3.73e-5	(-3.01e-5, 0.0001)
Σ_{44}	0.0310	(0.0175, 0.0499)

Notes. I estimate a vector autoregression with stochastic variance

$$Y_{t+1} = F + GY_t + \sigma_t \Sigma^{1/2} w_{t+1}$$

and restrictions: (1) $G_{21}/G_{11} = G_{22}/G_{12} = (G_{23} - 1)/G_{13} = (F_2 - \log \beta)/F_1 = \rho - 1$ and (2) $\alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)/2 + e_2'Ge_4 - e_2'\Sigma e_2/2 - [(\alpha - \rho)P + e_1(\alpha - 1)]'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]/2 + e_2'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)] - (\rho - 1)e_1'Ge_4 = 0$. Note that $\Sigma = HH'$, where H is from (2.5).
Vector $Y_t = (\log g_{t-1,t}, \ \log \pi_{t-1,t}, \ i_t^1, \ \sigma_t^2)'$ includes US consumption growth, inflation, one-period nominal yield, and stochastic variance.
To save space, I do not duplicate the symmetric entries of the matrix Σ. Sample period: second quarter of 1947 – fourth quarter of 2011. Quarterly.

Table 7
Global identification

	Identification "Fast Inflation"			Identification "Fast Consumption"	
Parameter	Estimate	Confidence interval, 95%	Parameter	Estimate	Confidence interval, 95%
H_{11}	0.0054	(0.0045, 0.0065)	H_{11}	0.0051	(0.0043, 0.0061)
H_{14}	-0.0011	(-0.0024, 0.0002)	H_{12}	0.0017	(0.0007, 0.0028)
H_{21}	0.0019	(0.0009, 0.0030)	H_{14}	-0.0011	(-0.0024, 0.0002)
H_{22}	0.0057	(0.0047, 0.0069)	H_{22}	0.0060	(0.0050, 0.0072)
H_{24}	0.0014	(0.0003, 0.0025)	H_{24}	0.0014	(0.0003, 0.0025)
H_{31}	0.0005	(0.0002, 0.0009)	H_{31}	0.0004	(0.0002, 0.0007)
H_{32}	0.0003	(-4.8e-5, 0.0007)	H_{32}	0.0004	(9.3e-5, 0.0008)
H_{33}	0.0015	(0.0012, 0.0017)	H_{33}	0.0015	(0.0012, 0.0017)
H_{34}	0.0002	(-0.0002, 0.0006)	H_{34}	0.0002	(-0.0002, 0.0006)
H_{44}	0.1747	(0.1324, 0.2233)	H_{44}	0.1747	(0.1324, 0.2233)

Notes. I identify structural shocks ε_{t+1} from the reduced form innovations w_{t+1}: $\Sigma^{1/2} w_{t+1} = H\varepsilon_{t+1}$. I consider two globally exactly identified models. Identification "Fast Inflation" is determined by the following zero restrictions: $H_{12} = H_{13} = H_{23} = H_{41} = H_{42} = H_{43} = 0$. Identification "Fast Consumption" is determined by the following zero restrictions: $H_{13} = H_{21} = H_{23} = H_{41} = H_{42} = H_{43} = 0$. Quarterly.

Table 8
Estimated FX cash flow process (identification "Fast Inflation")

Parameter	Basket "Low"	Basket "Intermediate"	Basket "High"
$\log \delta$	-0.0011	-0.0176	-0.0077
	(-0.0161, 0.0144)	(-0.0355, -0.0038)	(-0.0289, 0.0080)
μ_g	1.6033	-0.1360	-1.4884
	(0.9346, 2.2631)	(-0.7566, 0.4897)	(-2.2656, -0.7477)
μ_π	-0.5243	-2.7934	-1.9153
	(-0.9637, -0.1050)	(-3.2375, -2.3639)	(-2.4630, -1.3713)
μ_i	0.0698	2.4242	2.0582
	(-0.3352, 0.4634)	(2.0549, 2.8050)	(1.5984, 2.5358)
μ_σ	-0.0110	0.0012	-0.0020
	(-0.0207, -0.0017)	(-0.0099, 0.0090)	(-0.0185, 0.0081)
ξ_g	-0.0034	0.0072	0.0163
	(-0.0074, 0.0003)	(0.0036, 0.0110)	(0.0106, 0.0222)
ξ_π	-0.0030	0.0070	0.0157
	(-0.0086, 0.0023)	(0.0020, 0.0121)	(0.0101, 0.0217)
ξ_i	-0.0120	-0.0100	0.0064
	(-0.0152, -0.0089)	(-0.0128, -0.0072)	(0.0036, 0.0091)
ξ_σ	$-7.18 \cdot 10^{-5}$	0.0003	0.0055
	(-0.0099, 0.0104)	(-0.0085, 0.0088)	(-0.0053, 0.0169)

Notes. For each currency basket, I estimate the FX cash flow process:

$$\log \delta_{t,t+1} = \log \delta + \mu' Y_t + \sigma_t \xi' \varepsilon_{t+1} + \xi_v \sigma_t v_{t+1},$$

where $\mu = (\mu_g, \ \mu_\pi, \ \mu_i, \ \mu_\sigma)'$ and $\xi = (\xi_g, \ \xi_\pi, \ \xi_i \ \xi_\sigma)'$. Quarterly. There are 95% confidence intervals in the brackets.

Table 9
Estimated FX cash flow process (identification "Fast Consumption")

Parameter	Basket "Low"	Basket "Intermediate"	Basket "High"
$\log \delta$	-0.0007	-0.0173	-0.0082
	(-0.0152, 0.0149)	(-0.0363, -0.0035)	(-0.0305, 0.0076)
μ_g	1.6081	-0.1311	-1.4615
	(0.9475, 2.2560)	(-0.7724, 0.5232)	(-2.2080, -0.6957)
μ_π	-0.5177	-2.8045	-1.9180
	(-0.9744, -0.0887)	(-3.2364, -2.3927)	(-2.4375, -1.4041)
μ_i	0.0632	2.4308	2.0496
	(-0.3225, 0.4349)	(2.0674, 2.7973)	(1.5791, 2.5139)
μ_σ	-0.0109	0.0010	-0.0020
	(-0.0202, -0.0025)	(-0.0106, 0.0088)	(-0.0195, 0.0078)
ξ_g	-0.0022	0.0047	0.0106
	(-0.0068, 0.0023)	(0.0005, 0.0086)	(0.0054, 0.0157)
ξ_π	-0.0041	0.0088	0.0202
	(-0.0089, 0.0008)	(0.0043, 0.0136)	(0.0146, 0.0263)
ξ_i	-0.0120	-0.0100	0.0064
	(-0.0151, -0.0088)	(-0.0126, -0.0073)	(0.0036, 0.0091)
ξ_σ	$-1.6 \cdot 10^{-5}$	0.0003	0.0053
	(-0.0106, 0.0102)	(-0.0087, 0.0093)	(-0.0054, 0.0171)

Notes. For each currency basket, I estimate the FX cash flow process:

$$\log \delta_{t,t+1} = \log \delta + \mu' Y_t + \sigma_t \xi' \varepsilon_{t+1} + \xi_v \sigma_t v_{t+1},$$

where $\mu = (\mu_g, \ \mu_\pi, \ \mu_i, \ \mu_\sigma)'$ and $\xi = (\xi_g, \ \xi_\pi, \ \xi_i, \ \xi_\sigma)'$. Quarterly. There are 95% confidence intervals in the brackets.

Table 10
Parameters of the fixed point problem

Parameter	p_g	p_π	p_i	p_σ	b_0	b_1
Estimate	2.46	-0.29	24.26	0.01	-4e-4	0.9912
Conf inter	(1.34, 3.68)	(-1.12, 0.48)	(18.17, 30.60)	(-0.11, 0.13)	(-1e-3, 0)	(0.9901, 0.9933)

Notes. I solve the approximate equation:

$$\log u_t \approx b_0 + b_1 \log \mu_t(u_{t+1}g_{t+1})$$

The value function is $\log u_t = \log u + p_g \log g_{t-1,t} + p_\pi \log \pi_{t-1,t} + p_i i_t^1 + p_\sigma \sigma_{t,1}^2$. Quarterly.

Table 11
Parameters q

	Identification "Fast Inflation"		Identification "Fast Consumption"	
Parameter	Estimate	Confidence interval, 95%	Estimate	Confidence interval, 95%
q_g	-0.29	(-0.36, -0.22)	-0.26	(-0.34, -0.18)
q_π	-0.04	(-0.14, 0.04)	-0.13	(-0.22, -0.04)
q_i	-0.33	(-0.43, -0.25)	-0.33	(-0.43, -0.25)
q_σ	-0.03	(-0.23, 0.19)	-0.03	(-0.23, 0.19)

Notes. Vector q is the vector of loadings on the structural shocks $\sigma_t\varepsilon_{t+1}$ in the pricing kernel $\log m_{t,t+1}$:

$$\log m_{t,t+1} \quad = \quad \log m + \eta'Y_t + q'\sigma_t\varepsilon_{t+1}, \qquad (2.6)$$

where $q = H'((\alpha - \rho)P + e_1(\alpha - 1))$, $q = (q_g, \ q_\pi, \ q_i, \ q_\sigma)'$. Preference parameters: $\alpha = -9$, $\rho = 1/3$, $\beta = 0.9924$. Quarterly.

Table 12
One-period risk premia (identification "Fast Inflation")

|	Basket "Low"	Basket "Intermediate"	Basket "High"
Short-run risk	-0.0010	0.0021	0.0048
	(-0.0026, 0.0001)	(0.0008, 0.0042)	(0.0021, 0.0087)
Inflation risk	-0.0001	0.0003	0.0007
	(-0.0008, 0.0002)	(-0.0003, 0.0012)	(-0.0006, 0.0024)
Long-run risk	-0.0040	-0.0034	0.0022
	(-0.0072, -0.0019)	(-0.0061, -0.0016)	(0.0009, 0.0041)
Variance risk	$5.3 \cdot 10^{-5}$	$6.3 \cdot 10^{-5}$	0.0003
	(-0.0015, 0.0017)	(-0.0011, 0.0014)	(-0.0016, 0.0027)
Total	-0.0052	-0.0009	0.0079
	[0.23]	[0.19]	[0.2]
Data	-0.0063	-0.0018	0.0050

Notes: One period risk premia associated with multiple sources of risk. Stochastic variance σ_t^2 is set to be equal 1. Quarterly. I report p-values in the square brackets and 95% confidence intervals in the round brackets. The last row "Data" reports the level of the observed average excess returns.

Table 13
One-period risk premia (identification "Fast Consumption")

	Basket "Low"	Basket "Intermediate"	Basket "High"
Short-run risk	-0.0006	0.0012	0.0028
	(-0.0020, 0.0006)	(0.0001, 0.0028)	(0.0010, 0.0055)
Inflation risk	-0.0006	0.0012	0.0028
	(-0.0017, 0.0001)	(0.0003, 0.0027)	(0.0008, 0.0057)
Long-run risk	-0.0040	-0.0034	0.0022
	(-0.0069, -0.0019)	(-0.0060, -0.0016)	(0.0008, 0.0041)
Variance risk	$4.8 \cdot 10^{-5}$	$4.3 \cdot 10^{-5}$	0.0003
	(-0.0014, 0.0016)	(-0.0010, 0.0012)	(-0.0014, 0.0026)
Total	-0.0051	-0.0009	0.0080
	[0.22]	[0.19]	[0.11]
Data	-0.0063	-0.0018	0.0050

Notes: One period risk premia associated with multiple sources of risk. Stochastic variance σ_t^2 is set to be equal 1. Quarterly. I report p-values in the square brackets and 95% confidence intervals in the round brackets. The last row "Data" reports the level of the observed average excess returns.

Figure 1: **Dynamics of the model's states**
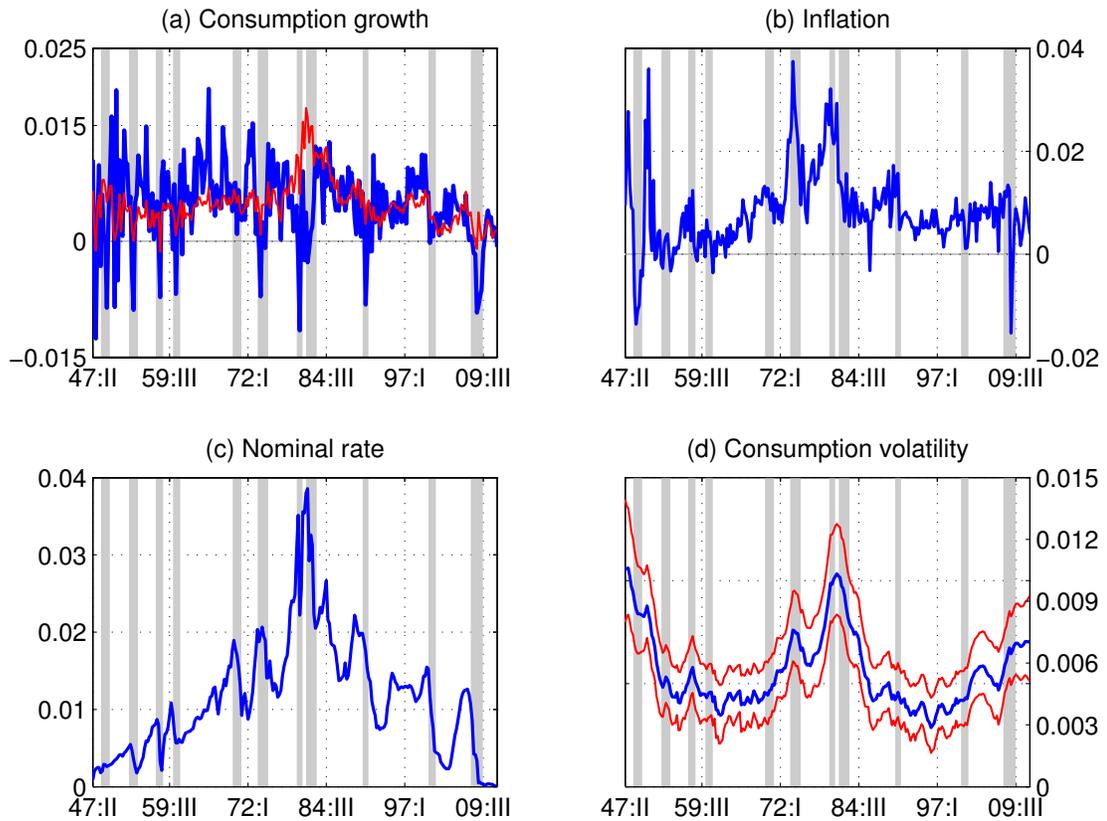
Panel (a) displays quarterly log consumption growth (thick blue line) and estimated expected consumption growth (thin red line). Panel (b) displays quarterly inflation. Panel (c) displays the 3-month nominal yield, quarterly. Panel (d) displays consumption volatility $\sqrt{\Sigma_{11}}\sigma_t$, quarterly. Blue line is the mean path of volatility, red lines correspond to the 95% confidence bounds. Grey bars are the NBER recessions.

Figure 2: **Shock-exposure elasticity (identification "Fast Inflation")**
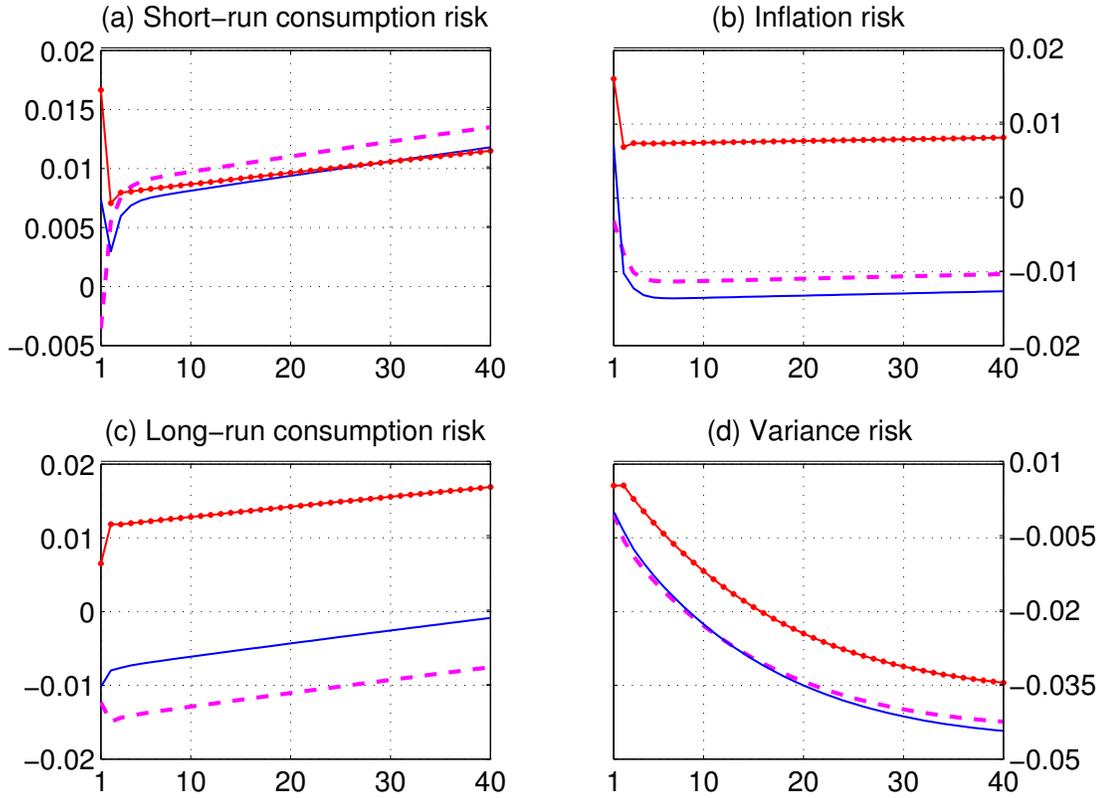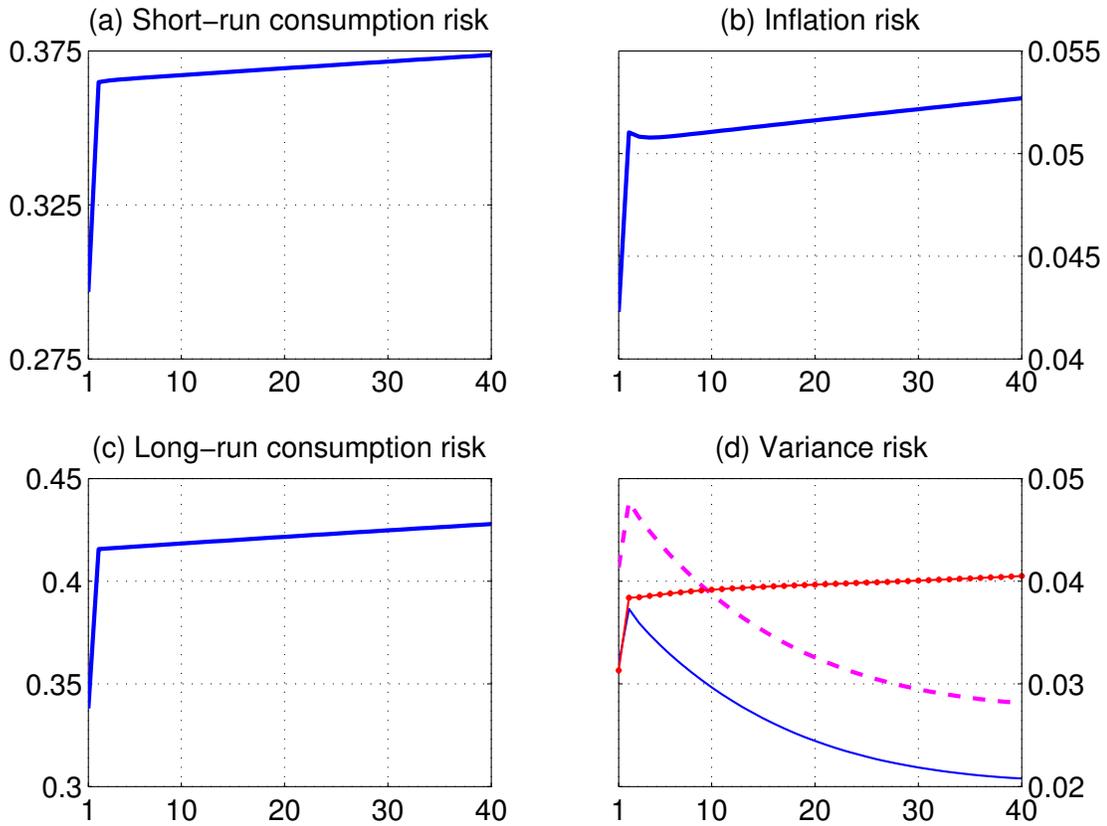

Panel (a) displays shock-exposure elasticity for the short-run consumption risk. Panel (b) displays shock-exposure elasticity for the inflation risk. Panel (c) displays shock-exposure elasticity for the long-run consumption risk. Panel (d) displays shock-exposure elasticity for the variance risk. Identification "Fast Inflation". Quarterly. The magenta dashed line is for the basket "Low", the blue solid line is for the basket "Intermediate", the red marked line is for the basket "High". The horizontal axes: from 1 quarter to 10 years.

Figure 3: **Shock-exposure elasticity (identification "Fast consumption")**
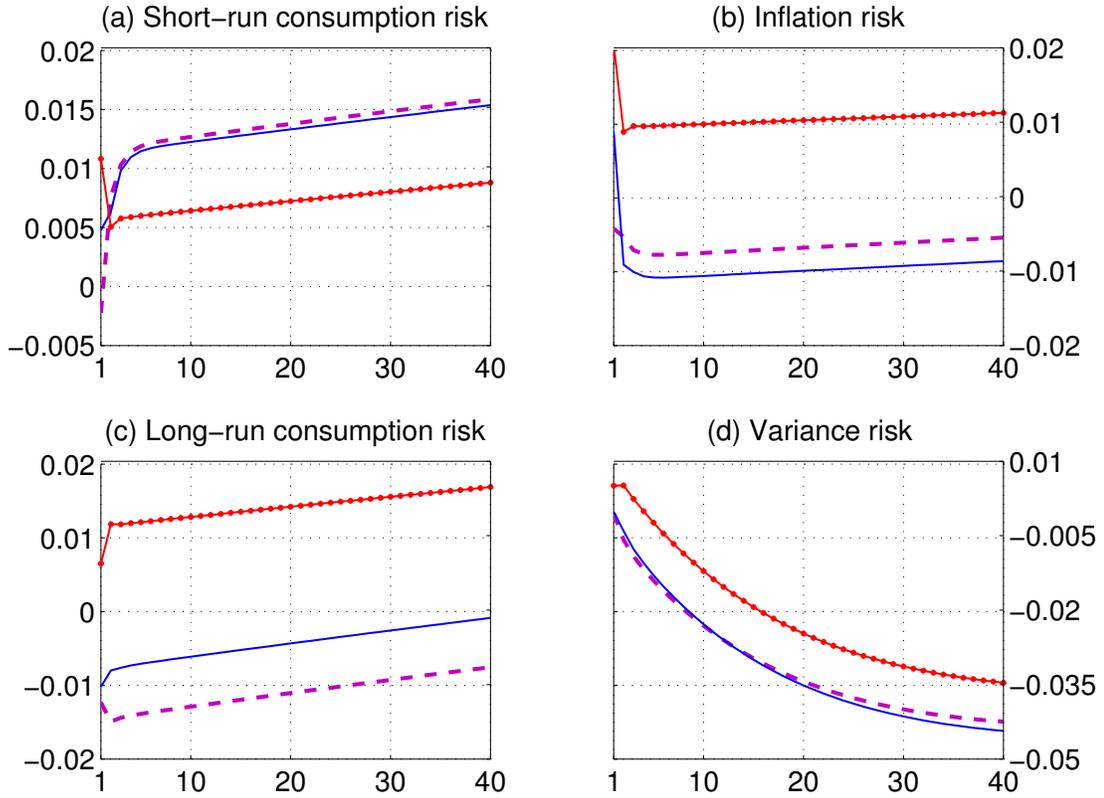
(a) Short–run consumption risk   (b) Inflation risk

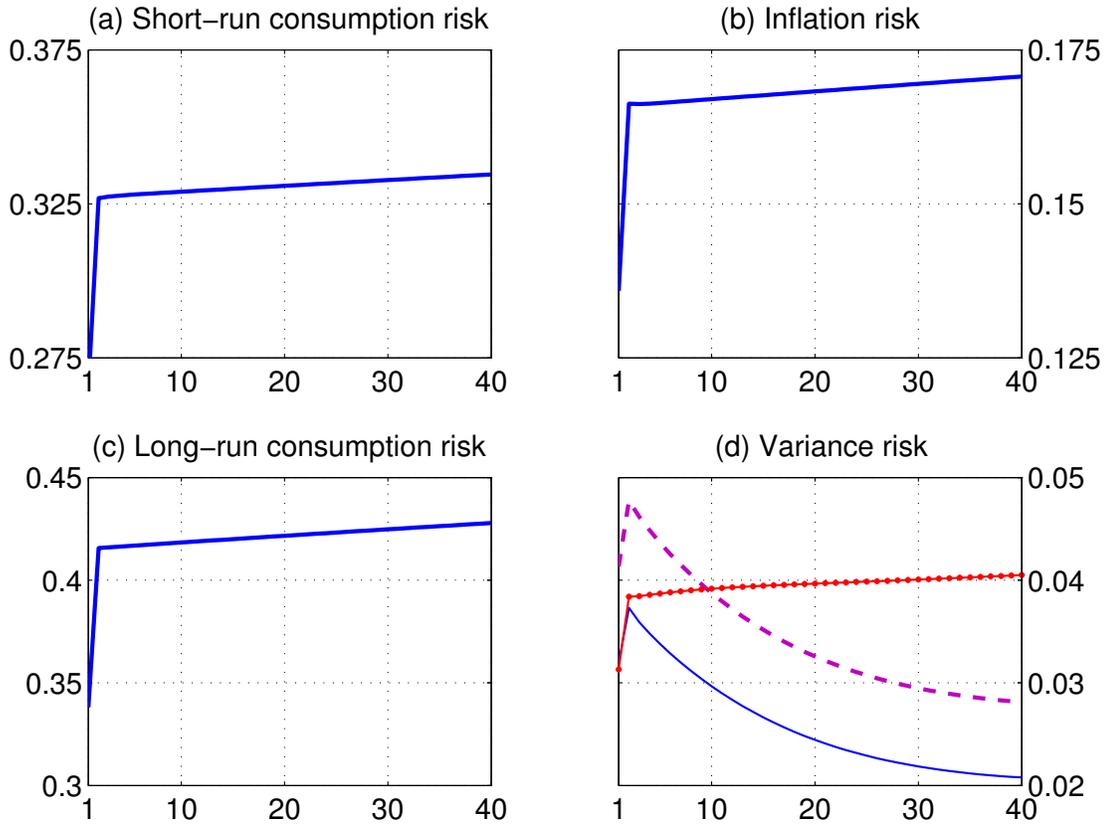(c) Long–run consumption risk   (d) Variance risk

Panel (a) displays shock-exposure elasticity for the short-run consumption risk. Panel (b) displays shock-exposure elasticity for the inflation risk. Panel (c) displays shock-exposure elasticity for the long-run consumption risk. Panel (d) displays shock-exposure elasticity for the variance risk. Identification "Fast Consumption". Quarterly. The magenta dashed line is for the basket "Low", the blue solid line is for the basket "Intermediate", the red marked line is for the basket "High". The horizontal axes: from 1 quarter to 10 years.

Figure 4: **Shock-price elasticity (identification "Fast Inflation")**

Panel (a) displays shock-price elasticity for the short-run consumption risk. Panel (b) displays shock-price elasticity for the inflation risk. Panel (c) displays shock-price elasticity for the long-run consumption risk. Panel (d) displays shock-price elasticity for the variance risk. The magenta dashed line is for the basket "Low", the blue solid line is for the basket "Intermediate", the red marked line is for the basket "High". The horizontal axes: from 1 quarter to 10 years. Identification "Fast Consumption". Quarterly.

Figure 5: **Shock-price elasticity (identification "Fast Consumption")**

Panel (a) displays shock-price elasticity for the short-run consumption shock. Panel (b) displays shock-price elasticity for the inflation shock. Panel (c) displays shock-price elasticity for the long-run consumption risk shock. Panel (d) displays shock-price elasticity for the variance shock. The magenta dashed line is for the basket "Low", the blue solid line is for the basket "Intermediate", the red marked line is for the basket "High". Identification "Fast Inflation". Quarterly.

A Appendix

A.1 Model's solution and pricing restrictions

In this Appendix, I derive the solution to my model. I briefly repeat the main building blocks for the ease of explicating.

The representative agent has recursive preferences

$$U_t = [(1-\beta)c_t^\rho + \beta\mu_t(U_{t+1})^\rho]^{1/\rho} \tag{A.17}$$

with the certainty equivalent function

$$\mu_t(U_{t+1}) = [E_t(U_{t+1}^\alpha)]^{1/\alpha}, \tag{A.18}$$

and preference parameters α (risk aversion is $1-\alpha$), β (subjective discount factor), and ρ ($1/(1-\rho)$ is the elasticity of intertemporal substitution).

The consumption growth process is described by a vector autoregressive system

$$Y_{t+1} = F + GY_t + H\sigma_t\varepsilon_{t+1}, \tag{A.19}$$

where $Y_{t+1} = (\log g_{t,t+1},\ \log \pi_{t,t+1},\ i_{t+1}^1,\ \sigma_{t+1}^2)'$.

To solve the model, I follow closely the solution method of Backus, Chernov, and Zin (2012). Since the utility U_t is determined by a constant elasticity of substitution recursion (A.17) and the certainty equivalent function is also homogenous of degree one, I scale (A.17) by consumption c_t:

$$u_t = [(1-\beta) + \beta\mu_t(u_{t+1}g_{t,t+1})^\rho]^{1/\rho}, \tag{A.20}$$

where $u_t = U_t/c_t$, and $g_{t,t+1} = c_{t+1}/c_t$.

The log pricing kernel under the recursive utility is

$$\log m_{t,t+1} = \log\beta + (\rho-1)\log g_{t,t+1} + (\alpha-\rho)(\log(u_{t+1}g_{t,t+1}) - \log\mu_t(u_{t+1}g_{t,t+1})) \tag{A.21}$$

Appendix A.5 of the NBER version of Backus, Chernov, and Zin (2012) provides the corresponding derivation.

To derive the pricing kernel, I need to solve the equation (A.20). I use a log-linear approximation of (A.20) to obtain a closed-form solution to the value function $\log u_t$ and to the pricing kernel:

$$\log u_t \approx b_0 + b_1 \log \mu_t(g_{t,t+1} u_{t+1}), \tag{A.22}$$

where

$$b_1 = \beta e^{\rho \log \mu} / [(1-\beta) + \beta e^{\rho \log \mu}], \tag{A.23}$$

$$b_0 = \rho^{-1} \log [(1-\beta) + \beta e^{\rho \log \mu}] - b_1 \log \mu. \tag{A.24}$$

The equation is exact if the elasticity of intertemporal substitution is equal to one. In such a case $b_0 = 0$ and $b_1 = \beta$. See Section III in Hansen, Heaton, and Li (2008) and Appendix A.7 in Backus, Chernov, and Zin (2012) for details about the log-linear approximation and its accuracy.

I guess that the solution to the equation (A.22) is an affine function of the four model's states:

$$\log u_t = \log u + P' Y_t, \tag{A.25}$$

where P is a vector of loadings $P = (p_g, \ p_\pi, \ p_i, \ p_\sigma)'$.

Next, I verify my guess. I compute the log of the certainty equivalent function

$$\log \mu_t(u_{t+1} g_{t,t+1}) = \underbrace{[\log u + e_1' F + P' F]}_{\log \mu} + [P'G + e_1'G]Y_t + \alpha[P+e_1]'\Sigma[P+e_1]\sigma_t^2/2 \tag{A.26}$$

where $\Sigma = HH'$ and e_1 is a coordinate vector with the first element equal to 1. Then I substitute (A.25) and (A.26) to the equation (A.22) and collect and match the corresponding terms. The equation (A.22) has a constant term and four variables, hence I obtain the system of five equations:

$$\log u = b_0 + b_1 \log u + b_1 e_1' F + b_1 P' F, \tag{A.27}$$

$$p_g = b_1(P + e_1)' G e_1 \tag{A.28}$$

$$p_\pi = b_1(P + e_1)' G e_2, \tag{A.29}$$

$$p_i = b_1(P + e_1)' G e_3, \tag{A.30}$$

$$p_\sigma = b_1(P + e_1)' G e_4 + \alpha b_1(P + e_1)' \Sigma(P + e_1)/2, \tag{A.31}$$

where e_i are the corresponding coordinate vectors.

Equations for p_g, p_π, and p_i are linear and therefore they result in unique solutions:

$$
\begin{aligned}
p_g &= A_g/B_g, \\
p_\pi &= A_\pi/B_\pi, \\
p_i &= A_i/B_i,
\end{aligned}
$$

where

$$
\begin{aligned}
A_g &= -(G_{11}b_1 - G_{11}G_{22}b1^2 + G_{12}G_{21}b_1^2 - G_{11}G_{33}b_1^2 + G_{13}G_{31}b_1^2 + G_{11}G_{22}G_{33}b_1^3 \\
&\quad - G_{11}G_{23}G_{32}b_1^3 - G_{12}G_{21}G_{33}b_1^3 + G_{12}G_{23}G_{31}b_1^3 + G_{13}G_{21}G_{32}b_1^3 - G_{13}G_{22}G_{31}b_1^2), \\
A_\pi &= -(G_{13}b_1 + G_{12}G_{23}b_1^2 - G_{13}G_{22}b_1^2), \\
A_i &= -(G_{12}b_1 + G_{13}G_{32}b_1^2 - G_{12}G_{33}b_1^2), \\
B_g &= B_\pi = B_i \\
&= G_{11}b_1 + G_{22}b_1 + G_{33}b_1 - G_{11}G_{22}b_1^2 + G_{12}G_{21}b_1^2 - G_{11}G_{33}b_1^2 + G_{13}G_{31}b_1^2 \\
&\quad - G_{22}G_{33}b_1^2 + G_{23}G_{32}b_1^2 + G_{11}G_{22}G_{33}b_1^3 - G_{11}G_{23}G_{32}b_1^3 - G_{12}G_{21}G_{33}b_1^3 \\
&\quad + G_{12}G_{23}G_{31}b_1^3 + G_{13}G_{21}G_{32}b_1^3 - G_{13}G_{22}G_{31}b_1^3 - 1
\end{aligned}
$$

The equation for p_σ is quadratic:

$$
A_\sigma p_\sigma^2 + B_\sigma p_\sigma + C_\sigma = 0,
$$

where

$$
\begin{aligned}
A_\sigma &= \alpha b_1 \Sigma_{44}/2, \\
B_\sigma &= \alpha b_1 (\Sigma_{34}p_i + \Sigma_{24}p_\pi + \Sigma_{14}(p_g + 1)) + b_1 G_{44} - 1, \\
C_\sigma &= \alpha b_1 ((p_g + 1)(\Sigma_{13}p_i + \Sigma_{12}p_\pi + \Sigma_{11}(p_g + 1)) + p_i(\Sigma_{33}p_i + \Sigma_{23}p_\pi + \Sigma_{13}(p_g + 1)) \\
&\quad + p_\pi(\Sigma_{23}p_i + \Sigma_{22}p_\pi + \Sigma_{12}(p_g + 1)))/2 + (b_1 p_g G_{14} + b_1 p_\pi G_{24} + b_1 p_i G_{34} + b_1 G_{14}).
\end{aligned}
$$

This equation has two real roots if its discriminant Discr $= (B_\sigma^2 - 4A_\sigma C_\sigma)$ is positive. Only one real root is good, however. It has to be selected based on the property of stochastic stability (Hansen (2012)),

$$
p_\sigma = \frac{-B_\sigma + \mathrm{sign}(B_\sigma)\mathrm{Discr}^{1/2}}{2A_\sigma}.
$$

Finally, $\log u$ follows as

$$\log u = [b_0 + b_1 e_1' F + b_1 P' F]/[1 - b_1].$$

I plug the solution $\log u_t$ into (A.21) and obtain the final expression for the pricing kernel

$$
\begin{aligned}
\log m_{t,t+1} &= [\log \beta + (\rho - 1)e_1' F] + (\rho - 1)e_1' GY_t - \alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)\sigma_t^2/2 \\
&+ [(\alpha - \rho)P + e_1(\alpha - 1)]' H\sigma_t \varepsilon_{t+1} \tag{A.32}
\end{aligned}
$$

or

$$\log m_{t,t+1} = \log m + \eta' Y_t + q'\sigma_t \varepsilon_{t+1},$$

where

$$
\begin{aligned}
\eta &= (\rho - 1)G'e_1 - \alpha(\alpha - \rho)e_4(P + e_1)'\Sigma(P + e_1)/2, \\
q &= H'[(\alpha - \rho)P + e_1(\alpha - 1)].
\end{aligned}
$$

Next, I derive a one-period real risk-free rate

$$
\begin{aligned}
r_{f,t}^1 &= -E_t(\log m_{t,t+1}) - Var_t(\log m_{t,t+1})/2 \\
&= -\log \beta - (\rho - 1)e_1' F - (\rho - 1)e_1' GY_t + \alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)\sigma_t^2/2 \\
&- [(\alpha - \rho)P + e_1(\alpha - 1)]'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]\sigma_t^2/2. \tag{A.33}
\end{aligned}
$$

Finally, the nominal one-period rate is

$$
\begin{aligned}
i_t^1 &= r_{f,t}^1 + E_t(\log \pi_{t,t+1}) - Var_t(\log \pi_{t,t+1})/2 + cov_t(\log m_{t,t+1}, \log \pi_{t,t+1}) \\
&= -\log \beta - (\rho - 1)e_1' F + e_2' F - (\rho - 1)e_1' GY_t + e_2' GY_t + \alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)\sigma_t^2/2 \\
&- [(\alpha - \rho)P + e_1(\alpha - 1)]'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]\sigma_t^2/2 - e_2'\Sigma e_2 \sigma_t^2/2 \\
&+ e_2'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]\sigma_t^2. \tag{A.34}
\end{aligned}
$$

Note that i_t^1 enters both the left-hand side and the right-hand side of (A.34), because the nominal yield i_t^1 is a part of the state-vector Y_t.

$$i_t^1 = A \log g_{t-1,t} + B \log \pi_{t-1,t} + Ci_t^1 + D\sigma_t^2 + E,$$

where

$$A = -\log \beta - (\rho - 1)e_1'F + e_2'F,$$
$$B = -(\rho - 1)e_1'Ge_1 + e_2'Ge_1$$
$$C = -(\rho - 1)e_1'Ge_2 + e_2'Ge_2,$$
$$D = -(\rho - 1)e_1'Ge_3 + e_2'Ge_3,$$
$$E = -[(\alpha - \rho)P + e_1(\alpha - 1)]'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]/2 - e_2'\Sigma e_2/2 + e_2'Ge_4$$
$$+e_2'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)] + \alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)/2$$
$$-(\rho - 1)e_1'Ge_4.$$

The expression (A.34) is not an equation which nails down the nominal rate, it is an identity. Therefore, to guarantee consistent pricing of the nominal yield, the following five restrictions must be satisfied:

$$A = 0, \quad B = 0, \quad C = 1, \quad D = 0, \quad E = 0.$$

Four restrictions $A = B = E = 0$, $C = 1$ are linear and can be written as

$$\frac{G_{21}}{G_{11}} = \frac{G_{22}}{G_{12}} = \frac{G_{23} - 1}{G_{13}} = \frac{F_2 - \log \beta}{F_1} = \rho - 1.$$

The other restriction is nonlinear and it involves the endogenous parameters p_g, p_π, p_i, and p_σ:

$$-[(\alpha - \rho)P + e_1(\alpha - 1)]'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)]/2 - e_2'\Sigma e_2/2 + e_2'Ge_4$$
$$+e_2'\Sigma[(\alpha - \rho)P + e_1(\alpha - 1)] + \alpha(\alpha - \rho)(P + e_1)'\Sigma(P + e_1)/2$$
$$-(\rho - 1)e_1'Ge_4 = 0. \tag{A.35}$$

A.2 Data description

Macro data come from the NIPA tables of the Bureau of Economic Analysis and CRSP. I use Table 2.1 (Personal income and its disposition), Table 2.3.4 (Personal indexes for personal consumption expenditures by major type of product) and Table 2.3.5 (Personal consumption expenditures by major type of product). I measure real consumption as per capita expenditure on non-durable goods and services. Non-durables and services is the sum of entries of the row 8 from Table 2.3.5 divided by entries of the row 8 from Table 2.3.4 and components of row 13 from Table 2.3.5 divided by components of row 13 from Table 2.3.4. I construct price index associated with personal consumption expenditures. Row 40 of the Table 2.3.1 provides population data.

Table 14 describes sources and availability of currency and fixed income data.

Table 14
Data description

Country	Data availability	FX data/Datastream Mnemonics	Source of term-structure data/Mnemonics
Australia	1987:I – 2011:IV	AUSTDOL and USDOLLR	Wright [1987:I – 2009:I] Bloomberg [2009:II – 2011:IV]: ADSW1, ..., ADSW10
Canada	1986:I – 2011:IV	CNDOLLR and USDOLLR	Wright [1986:I – 2009:I] Bloomberg [2009:II – 2011:IV]: CDSW1, ..., CDSW10
Denmark	1988:III – 2011:IV	DANISHK and USDOLLR	Bloomberg [1988:III – 2011:IV]: DKSW1, ..., DKSW10
Euro Area	2004:III – 2011:IV	EURSTER and USDOLLR	Bloomberg [2004:III – 2011:IV]: EUSW1V3, ..., EUSW10V3
Germany	1986:I – 1998:IV	DMARKER and USDOLLR	Wright [1986:I – 1998:IV]
Japan	1986:I – 2011:IV	JAPAYEN and USDOLLR	Wright [1986:I – 2009:I] Bloomberg [2009:II – 2011:IV]: JYSW1, ..., JYSW10
New Zealand	1990:I – 2011:IV	NZDOLLR and USDOLLR	Wright [1990:I – 2009:I] Bloomberg [2009:II – 2011:IV]: NDSW1, ..., NDSW10
Norway	1998:I – 2011:IV	NORKRON and USDOLLR	Wright [1998:I – 2009:I] Bloomberg [2009:II – 2011:IV]: NKSW1, ..., NKSW10
South Africa	1997:II – 2011:IV	COMRAND and USDOLLR	Bloomberg [1997:II – 2011:IV]: SASW1, ..., SASW10
Sweden	1992:IV – 2011:IV	SWEKRON and USDOLLR	Wright [1992:IV – 2009:I] Bloomberg [2009:II – 2011:IV]: SKSW1, ..., SKSW10
Switzerland	1988:I – 2011:IV	SWISSFR and USDOLLR	Wright [1986:I – 2009:I] Bloomberg [2009:II – 2011:IV]: SFSW1, ..., SFSW10
UK	1986:I – 2011:IV	USDOLLR	Wright [1986:I – 2009:I] Bloomberg [2009:II – 2011:IV]: BPSW1, ..., BPSW10

Notes: Data availability and data sources.

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A.3   Fixed point problem

In this Appendix, I sketch the fixed point problem embedded in the equation (A.22).

1. I guess b_0 and b_1 and solve equations (A.27)-(A.31).

2. I compute $\log \mu$ from (A.26). Next, I evaluate (A.23) and (A.24) to obtain b_0' and b_1':

$$
\begin{aligned}
b_1' &= \beta e^{\rho \log \mu} / [(1 - \beta) + \beta e^{\rho \log \mu}], \\
b_0' &= \rho^{-1} \log [(1 - \beta) + \beta e^{\rho \log \mu}] - b_1 \log \mu.
\end{aligned}
$$

3. If b_0' and b_1' are not close enough to the initial values of b_0 and b_1, I set $b_0 = b_0'$ and $b_1 = b_1'$ and return to Stage 2.

I iterate until I achieve convergence. I set the following convergence criterion: $(b_0 - b_0')^2 + (b_1 - b_1')^2 < 10^{-18}$.

A.4   Shock elasticity

In this section, I follow lead of Borovička and Hansen (2011) and derive the shock-exposure and the shock-price elasticity for the four sources of consumption risk ε_{t+1}.

Shock-exposure elasticity

The shock-exposure elasticity quantifies the term-structure of marginal quantities of risk. It depends on the functional form of the cash flow process and the evolution of the model's states.

The cash flow process is

$$
\log \delta_{t,t+1} = \log \delta + \mu' Y_t + \xi' \sigma_t \varepsilon_{t+1},
$$

where without loss of generality, I omit the idiosyncratic shock v_{t+1}.

The dynamics of the model's states is summarized in the vector autoregression:

$$
Y_{t+1} = F + G Y_t + H \sigma_t \varepsilon_{t+1}.
$$

The shock-exposure elasticity has the following mathematical representation

$$\ell_\delta(Y_t, \tau) = \left.\frac{\mathrm{d}\log E[\tilde{\delta}_{t,t+\tau}|Y_t]}{\mathrm{d}v}\right|_{v=0} = \alpha_h(Y_t) \cdot \tilde{E}_\delta(\varepsilon_{t+1}|Y_t),$$

where $\alpha_h(Y_t)$ is a vector which selects one source of risk ($\alpha_h(Y_t)\cdot\varepsilon_{t+1}$ has a unit standard deviation) and $\tilde{E}_\delta$ is an operator of the mathematical expectation under the change of measure represented by the random variable[23]

$$L^\delta_{t,\tau} = \frac{\delta_{t,t+1}E(\delta_{t,t+\tau}/\delta_{t,t+1}|Y_{t+1})}{E(\delta_{t,t+1}E(\delta_{t,t+\tau}/\delta_{t,t+1}|Y_{t+1})|Y_t)}.$$

I derive the shock exposure elasticity by using the multiplicative factorization of the multi-period cash flow and applying the law of iterated expectations a number of times.

First, I compute $L^\delta_{t,1}$:

$$L^\delta_{t,1} = \frac{\delta_{t,t+1}}{E(\delta_{t,t+1}|Y_t)} = \frac{\exp\left(\xi'\varepsilon_{t+1}\sigma_t\right)}{\exp\left(\xi'\xi\sigma_t^2/2\right)} = \frac{\exp\left(\tilde{e}'_\delta(0,Y_t)\varepsilon_{t+1}\right)}{\exp\left(\tilde{e}'_\delta(0,Y_t)\tilde{e}_\delta(0,Y_t)/2\right)},$$

where $\tilde{E}_\delta(\varepsilon_{t+1}|Y_t) = \tilde{e}_\delta(0,Y_t)$ and note that

$$\ell_\delta(Y_t, 1) = \alpha_h(Y_t) \cdot \xi\sigma_t.$$

Next, I use the law of iterated expectations

$$
\begin{aligned}
E(\delta_{t,t+\tau}|Y_t) &= E(\delta_{t,t+1}\delta_{t+1,t+2}\cdots\delta_{t+\tau-1,t+\tau}|Y_t) \\
&= E(\delta_{t,t+1}E(\delta_{t+1,t+2}\cdots E(\delta_{t+\tau-1,t+\tau}|Y_{t+\tau-1})|\cdots|Y_{t+1})|Y_t)
\end{aligned}
$$

and compute $E(\delta_{t,t+\tau}|Y_t)$ recursively.

I start with

$$
\begin{aligned}
E(\delta_{t+\tau-1,t+\tau}|Y_{t+\tau-1}) &= \exp\left(\log\delta + \mu Y_{t+\tau-1} + \xi'\xi\sigma_{t+\tau-1}^2/2\right) \\
&= \exp\left(\mathcal{A}_0(1) + \mathcal{A}_g(1)\log g_{t+\tau-2,t+\tau-1} + \mathcal{A}_\pi(1)\log \pi_{t+\tau-2,t+\tau-1} + \mathcal{A}_i(1)i^1_{t+\tau-1} + \mathcal{A}_\sigma(1)\sigma_{t+\tau-1}^2\right),
\end{aligned}
$$

[23]For example, $\alpha_h(Y_t) = (1,\ 0,\ 0,\ 0)'\sigma_t$, where $E(\sigma_t^2) = 1$, or $\alpha_h(Y_t) = (1,\ 0,\ 0,\ 0)'$ selects the short-run consumption shock. Other specifications of $\alpha_h(Y_t)$ are possible.

where

$$
\begin{aligned}
\mathcal{A}_0(1) &= \log \delta, \\
\mathcal{A}_g(1) &= \mu_g, \\
\mathcal{A}_\pi(1) &= \mu_\pi, \\
\mathcal{A}_i(1) &= \mu_i, \\
\mathcal{A}_\sigma(1) &= \mu_\sigma + \xi'\xi/2.
\end{aligned}
$$

Next, I compute

$$
\begin{aligned}
E(\delta_{t+\tau-2,t+\tau-1}E(\delta_{t+\tau-1,t+\tau}|Y_{t+\tau-1})|Y_{t+\tau-2}) &= \\
= \exp\big(\mathcal{A}_0(2) + \mathcal{A}_g(2)\log g_{t+\tau-3,t+\tau-2} &+ \mathcal{A}_\pi(2)\log \pi_{t+\tau-3,t+\tau-2} + \mathcal{A}_i(2)i^1_{t+\tau-2} + \mathcal{A}_\sigma(2)\sigma^2_{t+\tau-2}\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_0(2) &= \log \delta + \mathcal{A}_0(1) + [\mathcal{A}_g(1),\ \mathcal{A}_\pi(1),\ \mathcal{A}_i(1),\ \mathcal{A}_\sigma(1)]F \\
&= \log \delta + \mathcal{A}_0(1) + \mathcal{A}_g(1)F_1 + \mathcal{A}_\pi(1)F_2 + \mathcal{A}_i(1)F_3 + \mathcal{A}_\sigma(1)F_4, \\
\mathcal{A}_g(2) &= \mu_g + \mathcal{A}_g(1)G_{11} + \mathcal{A}_\pi(1)G_{21} + \mathcal{A}_i(1)G_{31} + \mathcal{A}_\sigma(1)G_{41}, \\
\mathcal{A}_\pi(2) &= \mu_\pi + \mathcal{A}_g(1)G_{12} + \mathcal{A}_\pi(1)G_{22} + \mathcal{A}_i(1)G_{32} + \mathcal{A}_\sigma(1)G_{42}, \\
\mathcal{A}_i(2) &= \mu_i + \mathcal{A}_g(1)G_{13} + \mathcal{A}_\pi(1)G_{23} + \mathcal{A}_i(1)G_{33} + \mathcal{A}_\sigma(1)G_{43}, \\
\mathcal{A}_\sigma(2) &= \mu_\sigma + \mathcal{A}_g(1)G_{14} + \mathcal{A}_\pi(1)G_{24} + \mathcal{A}_i(1)G_{34} + \mathcal{A}_\sigma(1)G_{44} \\
&+ 0.5([\mathcal{A}_g(1)\ \mathcal{A}_\pi(1)\ \mathcal{A}_i(1)\ \mathcal{A}_\sigma(1)]H + \xi')([\mathcal{A}_g(1)\ \mathcal{A}_\pi(1)\ \mathcal{A}_i(1)\ \mathcal{A}_\sigma(1)]H + \xi')'/2.
\end{aligned}
$$

Finally, for a generic τ,

$$
E(\delta_{t,t+\tau}|Y_t) = \exp\big(\mathcal{A}_0(\tau) + \mathcal{A}_g(\tau)\log g_{t-1,t} + \mathcal{A}_\pi(\tau)\log \pi_{t-1,t} + \mathcal{A}_i(\tau)i^1_t + \mathcal{A}_\sigma(\tau)\sigma^2_t\big),
$$

where the parameters of the conditional expectation are determined by the system of

difference equations:

$$
\begin{aligned}
\mathcal{A}_0(\tau) &= \log \delta + \mathcal{A}_0(\tau - 1) + [\mathcal{A}_g(\tau - 1),\ \mathcal{A}_\pi(\tau - 1),\ \mathcal{A}_i(\tau - 1),\ \mathcal{A}_\sigma(\tau - 1)]F, \\
\mathcal{A}_g(\tau) &= \mu_g + \mathcal{A}_g(\tau - 1)G_{11} + \mathcal{A}_\pi(\tau - 1)G_{21} + \mathcal{A}_i(\tau - 1)G_{31} + \mathcal{A}_\sigma(\tau - 1)G_{41}, \\
\mathcal{A}_\pi(\tau) &= \mu_\pi + \mathcal{A}_g(\tau - 1)G_{12} + \mathcal{A}_\pi(\tau - 1)G_{22} + \mathcal{A}_i(\tau - 1)G_{32} + \mathcal{A}_\sigma(\tau - 1)G_{42}, \\
\mathcal{A}_i(\tau) &= \mu_i + \mathcal{A}_g(\tau - 1)G_{13} + \mathcal{A}_\pi(\tau - 1)G_{23} + \mathcal{A}_i(\tau - 1)G_{33} + \mathcal{A}_\sigma(\tau - 1)G_{43}, \\
\mathcal{A}_\sigma(\tau) &= \mu_\sigma + \mathcal{A}_g(\tau - 1)G_{14} + \mathcal{A}_\pi(\tau - 1)G_{24} + \mathcal{A}_i(\tau - 1)G_{34} + \mathcal{A}_\sigma(\tau - 1)G_{44} \\
&\quad + 0.5([\mathcal{A}_g(\tau - 1)\ \mathcal{A}_\pi(\tau - 1)\ \mathcal{A}_i(\tau - 1)\ \mathcal{A}_\sigma(\tau - 1)]H + \xi') \\
&\qquad ([\mathcal{A}_g(\tau - 1)\ \mathcal{A}_\pi(\tau - 1)\ \mathcal{A}_i(\tau - 1)\ \mathcal{A}_\sigma(\tau - 1)]H + \xi')'/2.
\end{aligned}
$$

In this case, the random variable associated with the change of measure is

$$
L_{t,\tau}^\delta = \frac{\exp\left(\tilde{e}_\delta'(\tau - 1, Y_t)\varepsilon_{t+1}\right)}{\exp\left(0.5(\tilde{e}_\delta'(\tau - 1, Y_t)\tilde{e}_\delta(\tau - 1, Y_t))'\right)},
$$

where

$$
\tilde{e}_\delta(\tau - 1, Y_t) = ([\mathcal{A}_g(\tau - 1),\ \mathcal{A}_\pi(\tau - 1),\ \mathcal{A}_i(\tau - 1),\ \mathcal{A}_\sigma(\tau - 1)]H + \xi')'\sigma_t.
$$

The shock-exposure elasticity immediately follows

$$
\begin{aligned}
\ell_\delta(Y_t, \tau) &= \alpha_h(Y_t) \cdot \tilde{e}_\delta(\tau - 1, Y_t) \\
&= \alpha_h(Y_t) \cdot ([\mathcal{A}_g(\tau - 1),\ \mathcal{A}_\pi(\tau - 1),\ \mathcal{A}_i(\tau - 1),\ \mathcal{A}_\sigma(\tau - 1)]H + \xi')'\sigma_t.
\end{aligned}
$$

Shock-price elasticity

To compute the shock-price elasticity (4.16), I need to evaluate the following object

$$
\ell_v(Y_t, \tau) = \left. \frac{d \log E[\tilde{\delta}_{t,t+\tau} m_{t,t+\tau}|Y_t]}{dv} \right|_{v=0}
$$

which has a similar mathematical structure to the shock-exposure elasticity. Borovička and Hansen (2011) call this object the shock-value elasticity. The shock-price elasticity, $\ell_p(Y_t, \tau)$, follows by means of subtracting the shock-value elasticity from the shock-exposure elasticity:

$$
\ell_p(Y_t, \tau) = \ell_\delta(Y_t, \tau) - \ell_v(Y_t, \tau).
$$

The derivation of the shock-value elasticity mirrors one of the shock-exposure elasticity.

Therefore, the solution has a similar mathematical representation:

$$\ell_v(Y_t, \tau) = \alpha_v(Y_t) \cdot ([\mathcal{B}_g(\tau - 1),\ \mathcal{B}_\pi(\tau - 1),\ \mathcal{B}_i(\tau - 1),\ \mathcal{B}_\sigma(\tau - 1)]H + \xi' + q')'\sigma_t,$$

where $\mathcal{B}_g$, $\mathcal{B}_\pi$, $\mathcal{B}_i$, and $\mathcal{B}_\sigma$ solve the system of difference equations:

$$
\begin{aligned}
\mathcal{B}_0(\tau) &= \log \delta + \log m + \mathcal{B}_0(\tau - 1) + [\mathcal{B}_g(\tau - 1),\ \mathcal{B}_\pi(\tau - 1),\ \mathcal{B}_i(\tau - 1),\ \mathcal{B}_\sigma(\tau - 1)]F, \\
\mathcal{B}_g(\tau) &= \mu_g + \eta_g + \mathcal{B}_g(\tau - 1)G_{11} + \mathcal{B}_\pi(\tau - 1)G_{21} + \mathcal{B}_i(\tau - 1)G_{31} + \mathcal{B}_\sigma(\tau - 1)G_{41}, \\
\mathcal{B}_\pi(\tau) &= \mu_\pi + \eta_\pi + \mathcal{B}_g(\tau - 1)G_{12} + \mathcal{B}_\pi(\tau - 1)G_{22} + \mathcal{B}_i(\tau - 1)G_{32} + \mathcal{B}_\sigma(\tau - 1)G_{42}, \\
\mathcal{B}_i(\tau) &= \mu_i + \eta_i + \mathcal{B}_g(\tau - 1)G_{13} + \mathcal{B}_\pi(\tau - 1)G_{23} + \mathcal{B}_i(\tau - 1)G_{33} + \mathcal{B}_\sigma(\tau - 1)G_{43}, \\
\mathcal{B}_\sigma(\tau) &= \mu_\sigma + \eta_\sigma + \mathcal{B}_g(\tau - 1)G_{14} + \mathcal{B}_\pi(\tau - 1)G_{24} + \mathcal{B}_i(\tau - 1)G_{34} + \mathcal{B}_\sigma(\tau - 1)G_{44} \\
&\quad + 0.5(q' + \xi' + [\mathcal{B}_g(\tau - 1)\ \mathcal{B}_\pi(\tau - 1)\ \mathcal{B}_i(\tau - 1)\ \mathcal{B}_\sigma(t - 1)]H) \\
&\quad (q' + \xi' + [\mathcal{B}_g(\tau - 1)\ \mathcal{B}_\pi(\tau - 1)\ \mathcal{B}_i(\tau - 1)\ \mathcal{B}_\sigma(\tau - 1)]H)'.
\end{aligned}
$$

with the following initial conditions

$$
\begin{aligned}
\mathcal{B}_0(1) &= \log m + \log \delta, \\
\mathcal{B}_g(1) &= \mu_g + \eta_g, \\
\mathcal{B}_\pi(1) &= \mu_\pi + \eta_\pi, \\
\mathcal{B}_i(1) &= \mu_i + \eta_i, \\
\mathcal{B}_\sigma(1) &= \mu_\sigma + \eta_\sigma + (\xi + q)'(\xi + q)/2
\end{aligned}
$$

and

$$\ell_v(Y_t, 1) = \alpha_h(Y_t) \cdot (\xi + q)\sigma_t.$$